Introduction to Topology.

Exercise sheet 4: Connected and path connected topological spaces

Exercise 1. *Let X be a topological space, and let $A \subset B \subseteq \overline{A}$ be subspaces of X.*

i) Prove that if A is connected so is B.

ii) Show that $[a, b)$, $(a, b]$, $(-\infty, b]$ and $[b, \infty)$ are all connected in $\mathbb{R}$.

Exercise 2. *Let X be a topological space, and consider $I = [0, 1]$ as a subspace of $\mathbb{R}$ where $\mathbb{R}$ is given the standard topology. Furthermore, let the cone on X be quotient space $CX = X \times I/\sim$, where $\sim$ is the equivalence relation on the product space $X \times I$ given by $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Show that CX is path-connected.*

Exercise 3. *Let X, Y be a pair of topological spaces. We define the disjoint union $X \coprod Y$ as the topological spaces whose underlying set is the disjoint union of X and Y and whose opens V are of the form $V = U_X \coprod U_Y$ for $U_X \subseteq X$ and $U_Y \subseteq Y$ open sets. Prove the following:*

i) Show that $X \coprod Y$ defines a topological space.

ii) Show that that a given a topological space Z and a pair of continuous maps $f : X \to Z$ and $g : Y \to Z$ there exists a unique continuous map $f \coprod g : X \coprod Y \to Z$ that restricts to f and g at each factor.

iv) Generalise this exercise for infinite coproducts.

Exercise 4. *Let $x \in X$. We define C_x the connected component of X at x as the union of all connected sets containing x. Prove the following:*

i) *For every $x \in X$, the subset C_x is connected and closed.*

ii) *Show that given $x \neq y$ if $C_x \cap C_y \neq \emptyset$ then $C_x = C_y$.*

iii) *We define a relation $x \sim y$ if and only if $C_x = C_y$. Show that $\sim$ defines an equivalence relation and conclude that $\mathrm{Con}(X) = X/\sim = *$ if and only if X is connected.*

iv) *Show that the only connected subsets of $\mathbb{Q}$ are the singletons $\{q\}$ for $q \in \mathbb{Q}$. Conclude that X is not in general homemorphic to the disjoint union of its connected components.*

**)* *Show that if for every $x \in X$ there exists a connected neighbourhood of X then*

$$X \simeq \coprod_{[x] \in \mathrm{Con}(X)} C_x$$

Exercise 5. *Let $x \in X$. We define P_x (the path connected component of X at x) as the set of points $y \in X$ such that there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

- *Show that for every $x \in X$ we have $P_x \subseteq C_x$.*
- *Show that the relation $x \sim y$ if $y \in P_x$ defines an equivalence relation.*
- *Show that $\mathrm{PCon}(X) = X/\sim = *$ if and only if X is path connected.*

Exercise 6. *Let $f : X \to Y$ and $g : Y \to X$ be a pair of continuous maps and suppose that for every $x \in X$ there exists a path from x to $g \circ f(x)$. Show that if Y is path connected so is X.*

Exercise 7. *Show that a subspace of $\mathbb{R}$ is connected if and only if it is an interval.*