Introduction to Topology.

Exercise sheet 3: Connected and compact topological spaces

Exercise 1. *Let X be an a compact space, Y a Hausdorff space and $f : X \to Y$ a continuous map.*

i) Show that f is a closed map.

ii) Show that if f is surjective, then f is a quotient map.

iii) Show that if f is bijective, then f is a homeomorphism.

iv) Show that f is proper*, i.e., for each compact subset $K \subseteq Y$ then its preimage $f^{-1}(K)$ is compact.*

Exercise 2. *Let X be a topological space and let $K_1, K_2 \subseteq X$ be compact subsets. Show that if X is Hausdorff then $K_1 \cap K_2$ is compact.*

Exercise 3. *Let X be any topological space. We wish to construct a topological space $\hat{X}$ which we call the one point compactification of X. As a set we define $\hat{X} = X \cup \{\infty\}$. A subset $U \subseteq \hat{X}$ is said to be open if:*

- *We have that $\infty \notin U$ and U is open in X.*
- *We have that $\infty \in U$ and $U = X \setminus (CK) \cup \{\infty\}$ where $CK \subseteq X$ is compact and closed.*

Prove the following:

i) The collection of open sets above define a topology on $\hat{X}$.

ii) The space $\hat{X}$ is compact.

iii) The space $\hat{X}$ is Hausdorff if and only if X is.

iv) Show that we have a continuous open map $\iota : X \to \hat{X}$. In particular, $\hat{X} \setminus \{\infty\} \simeq X$.

v) Suppose that X is Hausdorf and let $x_0 \in X$, then it follows that $\hat{X} \setminus \{x_0\} \simeq X$.

vi) What happens if we one point compactify the natural numbers?

Exercise 4 ($\star$). *Let $n \geq 1$, prove that $\hat{\mathbb{R}}^n \simeq \mathbb{S}^n$. (Hint: Use the stereographic projection).*

Exercise 5. *Let $K_1 \supseteq K_2 \supseteq K_3 \cdots$ be a descending chain of nonempty, closed, compact sets. Then it follows that $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$.*