Introduction to Topology.

Exercise sheet 5: Homotopy theory

March 21, 2024

Exercise 1. *Let X be a topological space. Show that X is path-connected if and only if every two constant maps $c_1 : X \to X$ and $c_2 : X \to X$ are homotopic.*

Exercise 2. *Let X, Y and Z be topological spaces. Show that if $f : X \to Y$ is homotopic to $f' : X \to Y$ and $g : Y \to Z$ is homotopic to $g' : Y \to Z$, then $g \circ f : X \to Z$ is homotopic to $g' \circ f'$.*

Exercise 3. *Let S^2 be the 2-sphere considered to be a subspace of $\mathbb{R}^3$ with the standard topology. Show that $\pi_1(S^2 \setminus \{(0,0,1)\}, s)$, is the trivial group where $s = (0,0,-1)$. Can you generalise this for $n \geq 1$?*

Exercise 4. *Let X be a Hausdorff space, and let A be a subspace of X. Show that if A is a retract of X, then A is a closed subset of X.*

Exercise 5. *Let X be a topological space and let S^1 be the unit circle considered as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Show that if $f : X \to S^1$ is not surjective, then f is homotopic to a constant map (nullhomotopic).*

Exercise 6. *A topological space X is said to be contractible if the identity map of X, $\mathrm{id}_X : X \to X$ is nullhomotopic.*

Show that a topological space X is contractible if and only if X is homotopy equivalent to the one-point space.

Exercise 7 (*). *Given a topological space X we define a category $\Pi(X)$ as follows:*

- *The objects of* $\Pi(X)$ *are the elements of the set* X.
- *For every* $x, y \in X$ *we define* $\mathrm{Hom}_{\Pi(X)}(x, y)$ *to be the set of path homotopy classes of maps* $f : I \to X$ *such that* $f(0) = x$ *and* $f(1) = y$.
- *Given* $x, y, z \in X$ *we define a map*

  $$\mathrm{Hom}_{\Pi(X)}(x, y) \times \mathrm{Hom}_{\Pi(X)}(y, z) \to \mathrm{Hom}_{\Pi(X)}(x, z); \quad (f, g) \mapsto g * f$$

  given by concatenation of paths.

Show that this definition yields a category where every morphism is an isomorphism (this is called a groupoid[1]*). Show that* $\mathrm{Hom}_{\Pi(X)}(x, x) = \pi_1(X, x)$.

[1] The category $\Pi(X)$ is known as the fundamental groupoid of X