Introduction to Topology.

Exercise sheet 2: Constructing topological spaces

Exercise 1. *Let $\mathcal{B}$ be the collection of half-open intervals of the form $[a, b) = \{x \in \mathbb{R} |\ a \leq x < b\}$ in $\mathbb{R}$.*

i) Show that $\mathcal{B}$ defines a basis for a topology on $\mathbb{R}$.

ii) Find the closure of the set $(0, 1)$ in this topology.

Exercise 2. *Let*[1] $\mathrm{Spec}(\mathbb{Z}) = \{p \in \mathbb{Z} \mid p \text{ is a prime number}\} \cup \{0\}$. *For every $m \in \mathbb{Z}$ such that $m \neq 0, 1$ define subsets $D(m) \subseteq \mathrm{Spec}(\mathbb{Z})$ as follows*

$$D(m) = \{p \in \mathrm{Spec}(\mathbb{Z}) \mid p \text{ does not divide } m\}$$

Prove that:

i) The collection $D(m)$ defines a basis for a topology on $\mathrm{Spec}(\mathbb{Z})$.

ii) Show that $0 \in \mathrm{Spec}(\mathbb{Z})$ is a dense point. In other words, prove that $\overline{0} = \mathrm{Spec}(\mathbb{Z})$.

iii) Is $\mathrm{Spec}(\mathbb{Z})$ a Hausdorff space?

Exercise 3. *Let X be a topological space and let Y be a subspace of X. If A is a subset of Y , show that the subspace topology on A inherited from Y is equal to the subspace topology on A inherited from X.*

Exercise 4. *Let X and Y be topological spaces, and let A and B be subsets of X and Y, respectively. Show that the topology on $A \times B$ as a subspace of the product $X \times Y$ is equal to the product topology on $A \times B$ where A and B are given the subspace topology inherited from X and Y, respectively.*

[1] We will not view 1 as a prime number

Exercise 5. *Let* $\mathbb{Q}$ *be the set of rational numbers equipped with the subspace topology inherited from* $\mathbb{R}$ *show that the set*

$$A = \left\{x \in \mathbb{Q} \mid -\sqrt{5} < x < \sqrt{5}\right\}$$

is both open and closed in $\mathbb{Q}$. *Is* $\mathbb{Q}$ *connected?*

Exercise 6. *Let* X *and* Y *be two topological spaces, and let* $X \times Y$ *be given the product topology. Show that if* $f : X \to Y$ *is a continuous map, the subspace*

$$G = \{(x, y) \in X \times Y \mid y = f(x)\}$$

of $X \times Y$ *is homeomorphic to* X.

Exercise 7. *A* T_1 *space is one in which for every pair of points* $x \neq y$ *there is an open set containing* x *but not* y. *Prove that a space is* T_1 *if and only if every singleton set* $\{x\}$ *is closed. Prove that the only* T_1 *topology on a finite set is the discrete topology.*

Exercise 8. *Let* X *be a topological space and* $A \subseteq X$ *a dense (proper) subset of* X. *Show that given a quotient map* $\pi : X \to Y$ *the image of* A *under* π, $\pi(A)$ *is also dense in* Y. *Conclude that the quotient space* $\mathbb{R}/\mathbb{Q}$ *obtained by collapsing all rational numbers to a point cannot be Hausdorff.*