Introduction to Topology.

Exercise sheet 0: Metric spaces

Exercise 1. *Does the function* $d(x,y) = (x-y)^2$ *define a metric on* $X = \mathbb{R}$*?*

Exercise 2. *Let* $\mathcal{M}_{n\times n}(\mathbb{R})$ *denote the set of* $(n \times n)$*-matrices with real coefficients. Does the function*

$$d : \mathcal{M}_{n\times n}(\mathbb{R}) \times \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R};$$

$$(A,B) \longmapsto |\det(A-B)|$$

define a metric on $\mathcal{M}_{n\times n}(\mathbb{R})$*?*

Exercise 3. *Let* (X,d) *be a metric space. Show that the function*

$$\hat{d} : X \times X \longrightarrow \mathbb{R}; \quad (x,y) \longmapsto \tfrac{d(x,y)}{1+d(x,y)}$$

defines a metric on X.

Exercise 4. *Let* X *be a set and* $n \geq 1$. *We denote by* X^n *the* n*-th fold cartesian product of* X *with itself. We define a map*

$$\partial_X^n : X^n \times X^n \longrightarrow \mathbb{R}$$

by setting $\partial_X^n((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) = \{$*Number of indices* $1 \leq i \leq n$ *such that* $x_i \neq y_i\}$.

i) Show that ∂_X^n *defines a metric for* $n \geq 1$.

ii) Which metric does it give when $n = 1$*?*

iii) Describe the open balls in (X^n, ∂_X^n).

Exercise 5. *Given a set* X *and two metrics* d_1 *and* d_2 *we say that* d_1 *is equivalent to* d_2 *if there exists constants* $L, M \in \mathbb{R}$ *such that*

$$d_1(x,y) \leq L d_2(x,y) \quad \text{and} \quad d_2(x,y) \leq M d_1(x,y).$$

i) Show that if two metrics are equivalent then the open sets of (X, d_1) and the open sets of (X, d_2) coincide.

ii) Prove that the the euclidean metric and the taxicab metric on $\mathbb{R}^n$ are equivalent.

Exercise 6. *We say that two metric spaces (X, d_X), (Y, d_Y) are homeomorphic if there exists continuous maps $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (X, d_X)$ such that $g \circ f = \mathrm{id}_X$ is the identity map on X and $f \circ g = \mathrm{id}_Y$ is the identity map on Y.*

i) Show that if (X, d_1) and (X, d_2) have equivalent metrics then they must be homeomorphic.

ii) Let $\mathbb{Z} \subset \mathbb{R}$ and let $(\mathbb{Z}, d_{\mathbb{Z}})$ the metric space obtained by restricting the euclidean metric on $\mathbb{R}$ to $\mathbb{Z}$. Show that $(\mathbb{Z}, d_{\mathbb{Z}})$ is homeomorphic to $(\mathbb{Z}, \partial_{\mathbb{Z}})$ where the later denotes $\mathbb{Z}$ equipped with the discrete metric.

iii) Are the metrics $d_{\mathbb{Z}}$ and $\partial_{\mathbb{Z}}$ equivalent?

Exercise 7. *Let (X, d_X) be a metric space. A sequence of points $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X$ is said to converge to $x_\infty \in X$ if the following holds:*

- *For every $\varepsilon > 0$ there exists some n_0 such that for every $n \geq n_0$ we have that $d(x_\infty, x_n) < \varepsilon$.*

Show that:

i) If the limit of $\{x_n\}_{n\in\mathbb{N}}$ exists then it is unique.

ii) Given $x_\infty \in X$ such that $x_\infty \neq x_i$ for $i \in \mathbb{N}$ then if x_∞ is the limit of $\{x_n\}_{n\in\mathbb{N}}$ then it is a boundary point of $\{x_n\}_{n\in\mathbb{N}} \subseteq X$.

iii) Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence and consider $x_\infty \in X$ such that $x_i \neq x_\infty$ for $i \in \mathbb{N}$. Show that if x_∞ is a boundary point of $\{x_n\}_{n\in\mathbb{N}}$ then it must be the limit of the sequence.

iv) We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N > 0$ such that for every $n, m \geq N$ we have $d(x_m, x_n) < \varepsilon$. Show that a point $x_\infty \in X$ such that $x_i \neq x_\infty$ for $i \in \mathbb{N}$ is the limit of $\{x_n\}_{n\in\mathbb{N}}$ if and only if it is a boundary point of $\{x_n\}_{n\in\mathbb{N}} \subseteq X$.

Exercise 8. *Let (X, d) be a metric space and $A \subseteq X$ such that $A \neq \emptyset$. Show that for every $x \in X$ such that x is a boundary point of A then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_i \in A$ for each $i \in \mathbb{N}$ with limit point $a_\infty = x$.*