Introduction to Topology — Revision Lecture

1 Metric spaces

Definition (Metric space)**.** A *metric space* is a set X together with a function $d\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, z) \leq d(x, y) + d(y, z)$.

Example. The standard examples are:

(i) $\mathbb{R}^n$ with the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$;

(ii) any set X with the discrete metric $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise.

Definition (Open balls, open sets, neighbourhoods)**.** Let (X, d) be a metric space, $x \in X$, and $r > 0$. The *open ball* of radius r centred at x is

$$\mathrm{B}(x; r) = \{y \in X : d(x, y) < r\}.$$

A subset $U \subseteq X$ is *open* if for every $x \in U$ there is some $r > 0$ with $\mathrm{B}(x; r) \subseteq U$. A subset is *closed* if its complement is open. A *neighbourhood* of x is a set containing some open ball around x.

Theorem. *For any metric space (X, d):*

(i) *every open ball is open;*

(ii) *arbitrary unions of open sets are open;*

(iii) *finite intersections of open sets are open.*

Hence the open subsets of (X, d) form a topology, called the metric topology*.*

Definition (Continuity between metric spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is *continuous at $x \in X$* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon.$$

It is *continuous* if it is continuous at every point of X.

2 From metric spaces to topological spaces

Continuity of $f\colon (X, d_X) \to (Y, d_Y)$ can be phrased in three equivalent ways:

$$\varepsilon\text{–}\delta \iff \text{neighbourhood version} \iff \text{preimage of open} = \text{open version}..$$

The *third* formulation uses only the open sets, not the metric. This is the crucial leap: topology keeps exactly the part of metric-space theory needed to talk about continuity.

Definition (Topological space)**.** A *topology* on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

(T1) $\varnothing, X \in \mathcal{T}$;

(T2) $\mathcal{T}$ is closed under arbitrary unions;

(T3) $\mathcal{T}$ is closed under *finite* intersections.

Elements of $\mathcal{T}$ are called *open sets*. A *closed set* is the complement of an open set.

Example. Here are some examples of topological spaces:

1. Every metric space (X, d) carries a *metric topology*, generated by the open balls. Two equivalent metrics give the same topology.
2. The *discrete* topology $\mathcal{T} = \mathcal{P}(X)$ and the *indiscrete* topology $\mathcal{T} = \{\varnothing, X\}$ are the extreme cases.
3. The *finite complement topology* on an infinite set is not metrisable.

Definition (Continuity, homeomorphism)**.** A map $f\colon X \to Y$ between topological spaces is *continuous* if $f^{-1}(V)$ is open in X for every open $V \subseteq Y$. A *homeomorphism* is a continuous bijection whose inverse is also continuous.

Remark. A continuous bijection need not be a homeomorphism (counterexample: $[0, 1) \to S^1$, $t \mapsto e^{2\pi i t}$). This is why we insist on both directions.

Theorem. *For $f\colon X \to Y$, TFAE:*

(i) *f is continuous;*

(ii) *$f^{-1}(K)$ is closed in X for every closed $K \subseteq Y$;*

(iii) *for every $x \in X$ and every open neighbourhood V of $f(x)$, there is an open neighbourhood U of x with $f(U) \subseteq V$;*

Definition. For $A \subseteq X$, the *closure* $\overline{A}$ is the smallest closed set containing A; the *interior* $\mathrm{Int}(A)$ is the largest open set contained in A. We call A *dense* if $\overline{A} = X$.

Remark.

$$x \in \overline{A} \quad \iff \quad \text{every open neighbourhood } U \ni x \text{ meets } A.$$

Also, A is dense in X iff every non-empty open subset of X meets A.

3 Generating topologies: bases and subbases

It is useful to describe a topology by specifying a small family of open sets.

Definition (Basis)**.** A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(B1) every $x \in X$ lies in some $B \in \mathcal{B}$;

(B2) if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then there is $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The *topology generated by* $\mathcal{B}$ consists of all unions of elements of $\mathcal{B}$.

Definition (Subbasis)**.** A *subbasis* is any collection $\mathcal{S}$ of subsets of X. The finite intersections of elements of $\mathcal{S}$ form a basis; the resulting topology is the *topology generated by the subbasis* $\mathcal{S}$.

Example. The open balls $\{\mathrm{B}(x;r)\}$ form a basis for the metric topology. The half-infinite rays $\{(-\infty, a)\} \cup \{(b, \infty)\}$ form a subbasis for the standard topology on $\mathbb{R}$.

Theorem (Checking continuity on a basis)**.** *If $\mathcal{B}$ is a basis for the topology on Y, then $f\colon X \to Y$ is continuous iff $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.*

4 Constructing new spaces

Three universal constructions dominate the course.

Subspace topology

For $A \subseteq X$, declare $V \subseteq A$ open iff $V = A \cap U$ for some open $U \subseteq X$. The inclusion $i\colon A \hookrightarrow X$ is continuous, and a map $f\colon Z \to A$ is continuous iff $i \circ f\colon Z \to X$ is.

Product topology

On $X \times Y$, take as a basis the rectangles $U \times V$ with U open in X and V open in Y. The projections $\mathrm{pr}_1, \mathrm{pr}_2$ are continuous, and

$$f\colon Z \longrightarrow X \times Y \quad \text{is continuous} \quad \Longleftrightarrow \quad \mathrm{pr}_1 \circ f \text{ and } \mathrm{pr}_2 \circ f \text{ are continuous.}$$

Quotient topology

Let $\pi\colon X \twoheadrightarrow A$ be a surjection onto a set A. The *quotient topology* on A declares $V \subseteq A$ open iff $\pi^{-1}(V)$ is open in X. Then π is a *quotient map*, and

$$f\colon A \longrightarrow Y \text{ continuous} \quad \Longleftrightarrow \quad f \circ \pi\colon X \to Y \text{ continuous.}$$

Lemma. *A surjective continuous map $\pi\colon X \to Y$ is a quotient map if it is either open or closed.*

Example (Standard quotients)**.** $[0,1]/(0 \sim 1) \cong S^1$; $D^2/\partial D^2 \cong S^2$; $[0,1]^2$ with opposite edges identified (same direction) $\cong$ torus T^2.

5 Topological properties

Connectedness

Definition. A *separation* of X is a pair of disjoint non-empty open sets U, V with $X = U \cup V$. X is *connected* if no separation exists. Equivalently: the only subsets of X that are both open and closed are $\varnothing$ and X.

Theorem. (i) *The continuous image of a connected space is connected.*

(ii) *If $\{A_\lambda\}$ are connected subspaces of X with $\bigcap_\lambda A_\lambda \neq \varnothing$, then $\bigcup_\lambda A_\lambda$ is connected.*

(iii) *A finite product of connected spaces is connected.*

(iv) *$\mathbb{R}$ is connected; hence so is every interval and every $\mathbb{R}^n$.*

Corollary. *If X is connected and $f\colon X \to \mathbb{R}$ is continuous, then for all $a, b \in X$ with $f(a) < r < f(b)$ (or $f(b) < r < f(a)$), there exists $c \in X$ such that $f(c) = r$.*

Definition. X is *path-connected* if for every $x, y \in X$ there is a continuous $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.

Theorem. *Path-connected $\Rightarrow$ connected. The converse fails: the* topologist's sine curve *is connected but not path-connected.*

Theorem. *The continuous image of a path-connected space is path-connected. In particular, every interval in $\mathbb{R}$ is path-connected, and every convex subset of $\mathbb{R}^n$ is path-connected.*

Hausdorff

Definition. X is *Hausdorff* if any two distinct points can be separated by disjoint open neighbourhoods.

Theorem.

(i) *Every metric space is Hausdorff.*

(ii) *In a Hausdorff space, every singleton $\{x\}$ is closed.*

(iii) *X is Hausdorff $\iff$ the diagonal $\Delta \subseteq X \times X$ is closed.*

(iv) *A finite product of Hausdorff spaces is Hausdorff.*

Compactness

Definition. A *cover* of X is a family of subsets whose union is X. X is *compact* if every open cover admits a finite subcover.

Theorem. (i) *A closed subset of a compact space is compact.*

(ii) *A compact subset of a Hausdorff space is closed.*

(iii) *The continuous image of a compact space is compact.*

(iv) *A (finite) product of compact spaces is compact.*

Theorem (Heine–Borel)**.** *A subset of $\mathbb{R}^n$ (with the Euclidean topology) is compact iff it is closed and bounded.*

Corollary. *If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then f attains its supremum and infimum on X.*

Theorem. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Remark. Connectedness, path-connectedness, compactness, and Hausdorffness are all *preserved by homeomorphisms*. So if X has one of these properties and Y does not, then X and Y cannot be homeomorphic. For example: $[0,1]$ is compact while $(0,1)$ is not; $\mathbb{R}$ is connected while $\mathbb{R} \setminus \{0\}$ is not.

6 Homotopy of maps and of paths

Definition. Continuous maps $f_0, f_1 \colon X \to Y$ are *homotopic*, written $f_0 \simeq f_1$, if there is a continuous map $H \colon X \times [0,1] \to Y$ with

$$H(x,0) = f_0(x), \qquad H(x,1) = f_1(x) \qquad \text{for all } x \in X.$$

Lemma. *If $X = A \cup B$ with A and B both closed/open in X, and if $f \colon A \to Y$, $g \colon B \to Y$ are continuous with $f|_{A\cap B} = g|_{A\cap B}$, then the combined map $X \to Y$ is continuous.*

Theorem. *Homotopy is an equivalence relation on $C(X,Y)$.*

Definition. Two paths $f_0, f_1 \colon [0,1] \to X$ from x_0 to x_1 are *path-homotopic*, written $f_0 \simeq_p f_1$, if there is a homotopy H between them that fixes the endpoints at every stage: $H(0,t) = x_0$, $H(1,t) = x_1$ for all $t \in [0,1]$.

If a map $f \colon X \to Y$ is homotopic to a constant map, we say that f is *nullhomotopic*.
Path homotopy is also an equivalence relation; write $[f]$ for the path homotopy class.

Definition. For $f \colon x_0 \rightsquigarrow x_1$ and $g \colon x_1 \rightsquigarrow x_2$, define $f * g \colon [0,1] \to X$ by

$$(f*g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2}, \\ g(2s-1) & \frac{1}{2} \le s \le 1. \end{cases}$$

Theorem. *Writing c_x for the constant path at x and $\bar{f}(s) = f(1-s)$, we have*

- ***Associativity:*** $[(f*g)*h] = [f*(g*h)]$;
- ***Identities:*** $[c_{x_0} * f] = [f] = [f * c_{x_1}]$;
- ***Inverses:*** $[f * \bar{f}] = [c_{x_0}]$ *and* $[\bar{f} * f] = [c_{x_1}]$.

7 The fundamental group

Definition. Let (X, x_0) be a based space. A *loop* at x_0 is a path $f \colon [0,1] \to X$ with $f(0) = f(1) = x_0$. The *fundamental group* is

$$\pi_1(X, x_0) \;:=\; \{\text{loops at } x_0\}/\simeq_p,$$

with group operation induced by $*$.

Theorem. *$\pi_1(X, x_0)$ is indeed a group.*

Theorem. *If X is path-connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Definition. A based map $h \colon (X, x_0) \to (Y, y_0)$ induces

$$h_* \colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \qquad h_*[f] = [h \circ f].$$

Theorem.
- $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$;
- $(k \circ h)_* = k_* \circ h_*$.

Corollary. *A homeomorphism $h \colon X \to Y$ induces an isomorphism*

$$h_* \colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, h(x_0)).$$

So π_1 is a topological invariant.

Theorem. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Definition. A path-connected space X is *simply connected* if $\pi_1(X)$ is trivial.

8 Homotopy equivalence and deformation retracts

Definition. A continuous $f\colon X \to Y$ is a *homotopy equivalence* if there is $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We call X and Y *homotopy equivalent* and write $X \simeq Y$.

Definition. A subspace $A \subseteq X$ is a *retract* if there is a continuous $r\colon X \to A$ with $r|_A = \mathrm{id}_A$. It is a *deformation retract* if additionally $i \circ r \simeq \mathrm{id}_X$, where $i\colon A \hookrightarrow X$ is the inclusion.

Lemma. *If A is a retract of X and $x_0 \in A$, then the inclusion induces an injection $i_*\colon \pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$.*

Definition. Let X be a topological space, and let A be a subspace of X. A homotopy

$$H\colon X \times I \to X$$

is called a *deformation retraction* of X onto A if

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) \in A \qquad \text{for all } x \in X,$$

and

$$H(a, t) = a \qquad \text{for all } a \in A \text{ and all } t \in I.$$

In this case, we say that A is a *deformation retract* of X.

Theorem. *If A is a deformation retract of X, the inclusion induces an isomorphism $i_*\colon \pi_1(A, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$.*

Theorem. *A homotopy equivalence $f\colon X \to Y$ induces an isomorphism $f_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$.*

Definition. A space X is *contractible* if id_X is nullhomotopic. Equivalently, X is contractible iff $X \simeq \{*\}$.

Corollary. *If X is contractible, then $\pi_1(X, x_0)$ is trivial for every $x_0 \in X$.*

Example. $\mathbb{R}^n$ is contractible; the annulus $\simeq S^1$; the Möbius band $\simeq S^1$; any convex subset of $\mathbb{R}^n$ is contractible, hence simply connected.

9 Covering spaces

Definition. A continuous surjection $p\colon E \to B$ is a *covering map* if every $b \in B$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open subsets of E, each mapped homeomorphically onto U by p.

Example. $p\colon \mathbb{R} \to S^1$, $p(x) = (\cos 2\pi x, \sin 2\pi x)$, is a covering map. The fibre over each point is a copy of $\mathbb{Z}$.

Theorem. *Let $p\colon E \to B$ be a covering. Given a path $f\colon [0,1] \to B$ and $e_0 \in p^{-1}(f(0))$, there is a* unique *lift $\tilde{f}\colon [0,1] \to E$ with $p \circ \tilde{f} = f$ and $\tilde{f}(0) = e_0$.*

Theorem. *Path homotopies lift uniquely. More precisely: if $F\colon [0,1] \times [0,1] \to B$ is a homotopy of paths and $\tilde{F}(\cdot, 0)$ is a chosen lift of $F(\cdot, 0)$, there is a unique lift $\tilde{F}$ of F extending it. Consequently, endpoints of lifted path-homotopic paths agree.*

Theorem. *Let $p\colon E \to B$ be a covering, with E path-connected, and fix $e_0 \in p^{-1}(b_0)$. The map*

$$\Phi\colon \pi_1(B, b_0) \longrightarrow p^{-1}(b_0), \qquad \Phi([f]) = \tilde{f}(1)$$

(where $\tilde{f}$ is the unique lift of f starting at e_0) is a well-defined surjection*, and it is a bijection if E is simply connected.*

10 The fundamental group of the circle

Theorem. $\pi_1(S^1, s_0) \cong \mathbb{Z}$, *generated by the loop that wraps once around* S^1.