Question 1 Let $X = \{a, b, c, d\}$ be a topological space with topology

$$\mathcal{T}_X = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, X\},$$

and let $Y = \{0, 1\}$ be a topological space with topology

$$\mathcal{T}_Y = \{\varnothing, \{1\}, Y\}.$$

Which one of the following maps $f\colon X \to Y$ is *not* continuous? Justify your answer.

(A) $f(a) = 1,\ f(b) = 1,\ f(c) = 0,\ f(d) = 0$

(B) $f(a) = 0,\ f(b) = 1,\ f(c) = 1,\ f(d) = 1$

(C) $f(a) = 1,\ f(b) = 1,\ f(c) = 1,\ f(d) = 0$

(D) $f(a) = 0,\ f(b) = 0,\ f(c) = 0,\ f(d) = 0$

Solution The open sets of Y are $\varnothing$, $\{1\}$, and Y. Since $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are always open, continuity of f reduces to the condition that $f^{-1}(\{1\})$ is open in X. We compute:

$$\begin{aligned}
&\text{(A) } f^{-1}(\{1\}) = \{a, b\}\\
&\text{(B) } f^{-1}(\{1\}) = \{b, c, d\}\\
&\text{(C) } f^{-1}(\{1\}) = \{a, b, c\}\\
&\text{(D) } f^{-1}(\{1\}) = \varnothing.
\end{aligned}$$

The sets $\{a, b\}$, $\{a, b, c\}$, and $\varnothing$ are all open in $\mathcal{T}_X$, so (A), (C), (D) are continuous. The set $\{b, c, d\}$ is *not* open in $\mathcal{T}_X$, so **(B) is not continuous**.

Question 2 Let $X = \{p, q, r, s, t\}$, and let

$$\mathcal{T} = \{\varnothing, \{q\}, \{p, q\}, \{q, r\}, \{p, q, r\}, \{q, s, t\}, \{p, q, s, t\}, \{q, r, s, t\}, X\}$$

be a topology on X. Which one of the following subsets of X is *not* dense in X? Justify your answer.

(A) $\{p, q, t\}$

(B) $\{q, s\}$

(C) $\{p, r, s\}$

(D) $\{q, r, t\}$

Solution A subset $A \subseteq X$ is dense if and only if $A \cap U \neq \varnothing$ for every nonempty open set $U \in \mathcal{T}$. Since every nonempty open set in $\mathcal{T}$ contains q, this condition is equivalent to $q \in A$. Therefore, by inspection, **(C)** is not dense.
Alternatively, one may compute the closed sets:

$$\varnothing,\ \{p\},\ \{r\},\ \{p, r\},\ \{s, t\},\ \{p, s, t\},\ \{r, s, t\},\ \{p, r, s, t\},\ X,$$

and note that A is dense if and only if $\overline{A} = X$, i.e., A is not contained in any proper closed set. Since $\{p, r, s\} \subseteq \{p, r, s, t\}$ and $\{p, r, s, t\}$ is a proper closed set, **(C)** is not dense.

Question 3 Let X and Y be two topological spaces, and let $f\colon X \to Y$ be a continuous bijection. Show that f is open if and only if f is closed.

Solution Since f is a bijection, for any subset $S \subseteq X$ we have

$$f(X \setminus S) = Y \setminus f(S).$$

($\Rightarrow$) Suppose f is open. Let $C \subseteq X$ be closed. Then $X \setminus C$ is open, so $f(X \setminus C) = Y \setminus f(C)$ is open in Y. Hence $f(C)$ is closed. Since C was arbitrary, f is closed.

($\Leftarrow$) Suppose f is closed. Let $U \subseteq X$ be open. Then $X \setminus U$ is closed, so $f(X \setminus U) = Y \setminus f(U)$ is closed in Y. Hence $f(U)$ is open. Since U was arbitrary, f is open.

Question 4 For each of the following statements, either prove that it is true, or show that it is false by giving a concrete counterexample.

(a) If X has the discrete topology, then every subspace of X has the discrete topology.

(b) If X is a topological space that does not have the discrete topology, then no subspace of X has the discrete topology.

(c) If $f\colon X \to Y$ is a surjective continuous function between topological spaces X and Y, and X is Hausdorff, then Y is Hausdorff as well.

Solution **(a) True.** Let $A \subseteq X$ carry the subspace topology. For any $S \subseteq A$ we have $S = A \cap S$. Since X has the discrete topology, S is open in X, so S is open in the subspace topology on A. Hence every subset of A is open, i.e. A has the discrete topology.

(b) False. For example, let $X = \mathbb{R}$ with the standard topology, which is not discrete. The subspace $\{0\} \subseteq \mathbb{R}$ has the discrete topology (every singleton is trivially discrete).

(c) False. For example, let $X = \{0, 1\}$ with the discrete topology (which is Hausdorff), and let $Y = \{0, 1\}$ with the indiscrete topology $\{\varnothing, Y\}$. The identity map $f = \mathrm{id}\colon X \to Y$ is a continuous surjection, and since Y carries the indiscrete topology, it is not Hausdorff.

Question 5 A continuous map $f\colon A \to B$ is called *proper* if for every compact subset $K \subseteq B$, the preimage $f^{-1}(K)$ is compact in A.

Assume $X \neq \varnothing$. Prove that the projection map

$$\pi_X\colon X \times Y \to X, \qquad (x, y) \mapsto x,$$

is proper if and only if Y is compact.

Solution ($\Rightarrow$) Suppose π_X is proper. Fix any $x_0 \in X$ (possible since $X \neq \varnothing$). The singleton $\{x_0\} \subseteq X$ is compact, so

$$\pi_X^{-1}(\{x_0\}) = \{x_0\} \times Y$$

is compact in $X \times Y$. The map $Y \to \{x_0\} \times Y$, $y \mapsto (x_0, y)$, is a homeomorphism, so Y is compact.

($\Leftarrow$) Suppose Y is compact. Let $K \subseteq X$ be compact. Then

$$\pi_X^{-1}(K) = K \times Y,$$

which is compact, since the (finite) product of compact spaces is compact. Hence π_X is proper.

Question 6 The *cone* on a topological space X is the quotient space

$$CX = (X \times [0,1]) \,/\, (X \times \{1\}),$$

where $X \times \{1\}$ is collapsed to a single point, called the *apex* of CX. Show that CX is path-connected for any topological space $X \neq \varnothing$.

Solution Let $* = [(x,1)]$ denote the apex of CX; this is independent of x. We show that every point of CX can be connected to $*$ by a path, which implies CX is path-connected (any two points $p, q \in CX$ are connected by concatenating a path from p to $*$ with the reverse of a path from q to $*$).
Let $[(x,t)] \in CX$ be any point. Define $\gamma\colon [0,1] \to CX$ by

$$\gamma(s) = [(x,\, t + s(1-t))].$$

This is well-defined: for all $s \in [0,1]$ we have $t + s(1-t) \in [0,1]$, and the map is the composition of the continuous map $s \mapsto (x, t+s(1-t))$ into $X \times [0,1]$ with the quotient map, hence continuous. Moreover $\gamma(0) = [(x,t)]$ and $\gamma(1) = [(x,1)] = *$. Hence $[(x,t)]$ and $*$ are connected by a path, as required.

Question 7 Let $\{0,1\}$ denote the two-point space with the discrete topology. Show that a topological space X is connected if and only if every continuous map $f\colon X \to \{0,1\}$ is constant.

Solution ($\Rightarrow$) Suppose X is connected and let $f\colon X \to \{0,1\}$ be continuous. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are disjoint open sets whose union is X. Since X is connected, one of them must be empty, so f is constant.

($\Leftarrow$) We prove the contrapositive. Suppose X is disconnected, so there exist nonempty disjoint open sets $U, V \subseteq X$ with $U \cup V = X$. Define $f\colon X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & x \in U, \\ 1 & x \in V. \end{cases}$$

Since $f^{-1}(\{0\}) = U$ and $f^{-1}(\{1\}) = V$ are both open, f is continuous. Since U and V

are both nonempty, f is not constant.

Question 8 Let X and Y be topological spaces.

(a) Suppose Y is Hausdorff. Show that the diagonal

$$\Delta_Y = \{(y, y) \mid y \in Y\} \subseteq Y \times Y$$

is closed in $Y \times Y$.

(b) Let Y be compact Hausdorff and let $f\colon X \to Y$ be a function. Show that f is continuous if and only if the graph of f,

$$G_f = \{(x, y) \in X \times Y \mid y = f(x)\},$$

is closed in $X \times Y$.

Hint: You may use, without proof, that the projection $\pi_X\colon X \times Y \to X$ is a closed map when Y is compact.

Solution **(a)** We show the complement $Y \times Y \setminus \Delta_Y$ is open. Let $(x, y) \in Y \times Y$ with $x \neq y$. Since Y is Hausdorff, there exist disjoint open sets $U \ni x$ and $V \ni y$. Then $U \times V$ is an open neighbourhood of (x, y) with $U \times V \subseteq Y \times Y \setminus \Delta_Y$ (if $(z, z) \in U \times V$ then $z \in U \cap V = \varnothing$, a contradiction). Hence the complement is open and Δ_Y is closed.

(b)

($\Rightarrow$) Suppose f is continuous. Consider the map

$$F\colon X \times Y \to Y \times Y, \qquad F(x, y) = (f(x), y).$$

Since f and id_Y are continuous, F is continuous. By part (a), Δ_Y is closed in $Y \times Y$. Moreover

$$G_f = F^{-1}(\Delta_Y),$$

since $(x, y) \in F^{-1}(\Delta_Y) \iff (f(x), y) \in \Delta_Y \iff f(x) = y$. Hence G_f is closed.

($\Leftarrow$) Suppose G_f is closed in $X \times Y$. Let $C \subseteq Y$ be closed. Then $X \times C$ is closed in $X \times Y$, so

$$G_f \cap (X \times C)$$

is closed in $X \times Y$. Since Y is compact, π_X is a closed map, so

$$\pi_X(G_f \cap (X \times C))$$

is closed in X. We claim this equals $f^{-1}(C)$: indeed,

$$\begin{aligned} x \in \pi_X(G_f \cap (X \times C)) &\iff \exists\, y \in C \text{ such that } (x, y) \in G_f \\ &\iff f(x) \in C \\ &\iff x \in f^{-1}(C). \end{aligned}$$

Hence $f^{-1}(C)$ is closed for every closed $C \subseteq Y$, so f is continuous.

Question 9 Suppose that Y is path-connected but that X is not path-connected. Show that there do not exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X.

Solution Suppose for contradiction that such f and g exist, with homotopy $H\colon X \times I \to X$ satisfying $H(x,0) = g(f(x))$ and $H(x,1) = x$ for all $x \in X$.
Since Y is path-connected and $g\colon Y \to X$ is continuous, $g(Y)$ is path-connected: any path α in Y between two points composes with g to give a path in $g(Y)$.
Now let $x_0, x_1 \in X$. The homotopy H provides paths $t \mapsto H(x_i, t)$ from $g(f(x_i))$ to x_i for $i = 0, 1$. Since $g(f(x_0)), g(f(x_1)) \in g(Y)$ and $g(Y)$ is path-connected, there is also a path from $g(f(x_0))$ to $g(f(x_1))$. Reversing the first homotopy path and concatenating the three paths in order gives a path from x_0 to x_1. Since x_0, x_1 were arbitrary, X is path-connected, a contradiction.

Question 10 Let X be the quotient space $X = ([0,1] \times [-1,1])/\!\sim$, where $\sim$ is the equivalence relation on $[0,1] \times [-1,1]$ given by

$$(0,t) \sim (1,-t) \qquad \text{for all } t \in [-1,1].$$

Show that $Y = \{[(s,0)] \mid 0 \le s \le 1\}$ is a deformation retract of X, and use this to determine the fundamental group of X.

Solution Let $q\colon [0,1] \times [-1,1] \to X$ be the quotient map. Define

$$H\colon X \times I \to X, \qquad H([(s,t)], r) = [(s, (1-r)t)].$$

This is well-defined: if $(0,t) \sim (1,-t)$, then

$$(0,(1-r)t) \sim (1,-(1-r)t).$$

The map is continuous by the universal property of the quotient topology, since the composite

$$([0,1] \times [-1,1]) \times I \xrightarrow{(s,t,r)\mapsto(s,(1-r)t)} [0,1] \times [-1,1] \xrightarrow{q} X$$

is continuous and is constant on the equivalence classes defining X.
Moreover,

$$H([(s,t)],0) = [(s,t)], \qquad H([(s,t)],1) = [(s,0)],$$

and if $[(s,0)] \in Y$, then

$$H([(s,0)], r) = [(s,0)]$$

for all $r \in I$. Hence H is a deformation retraction of X onto Y. Finally, $Y \cong S^1$, so

$$\pi_1(X) \cong \pi_1(Y) \cong \pi_1(S^1) \cong \mathbb{Z}.$$