TMA4190 - Introduction to Topology
Mid semester test

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Exercise 1. Define the notion of a topological space and provide the following examples:

(a) Three different topologies on the set $X = \{a, b, c, d\}$.

(b) A topological space which is a metric space.

(c) A topological space whose topology does not come from a metric space (remember to provide a proof of your claim).

Exercise 2. Which of the following statements is **not** true? Select one alternative, and justify your answer.

☐ The set $\{a, b\}$ equipped with the discrete topology is disconnected.

☐ Every homeomorphism is also an open and closed map.

☐ Every continuous map from a compact space to a Hausdorff space is a closed map.

☐ Every Hausdorff space is compact.

Exercise 3. Let X be a topological space and $A \subseteq X$.

(a) Define what it means for A to be:

(i) open in X,

(ii) closed in X,

(iii) dense in X.

(b) Prove that $\mathrm{Int}(X \setminus A) = X \setminus \overline{A}$ for any subset A of a topological space X.

(c) Give an example of a subset $A \subseteq \mathbb{R}$ (with the usual topology) such that the sets A, $\overline{A}$, and $\mathrm{Int}(A)$ are all different.

Exercise 4. Let $X = \{a, b, c, d, e, f\}$, and let

$$\mathcal{T} = \{\varnothing, \{a, b\}, \{a, b, c, e, f\}, \{a, b, d\}, \{c, d, e, f\}, \{c, e, f\}, \{d\}, X\}$$

be a topology on X. Determine the subspace topology on the subset

$$A = \{b, d, e\}.$$

Is A Hausdorff? Justify your answer.

Exercise 5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a map and define the *graph* of f by

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} \subseteq X \times Y.$$

Equip Γ_f with the subspace topology induced by the product topology on $X \times Y$. Show that f is continuous if and only if the map

$$g : X \to \Gamma_f, \qquad g(x) = (x, f(x)),$$

is a homeomorphism.

Exercise 6. Let X be a space.

(a) Prove that the union of any finite number of compact subsets of X is a compact subset of X.

(b) If X is Hausdorff, prove that the intersection of any collection of compact subsets of X is a compact subset of X.

Exercise 7. Let X be a topological space and let $A \subseteq B \subseteq X$. In this exercise, $\mathrm{cl}_X(-)$ denotes closure taken in X, and $\mathrm{cl}_B(-)$ denotes closure taken in the subspace B (with the subspace topology).

(a) Prove the identity

$$\mathrm{cl}_B(A) = B \cap \mathrm{cl}_X(A).$$

(b) Deduce that A is dense in B (i.e. $\mathrm{cl}_B(A) = B$) if and only if $\mathrm{cl}_X(A) \supseteq B$.

(c) Suppose that A is dense in B and that B is dense in X. Show that A is dense in X.

Exercise 8. Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions (where $\mathbb{R}$ has the usual topology).

(a) Show that the set $\{x \in X : f(x) < g(x)\}$ is open in X.

(b) Define $h : X \to \mathbb{R}$ by $h(x) = \min\{f(x), g(x)\}$. Prove that h is continuous.

Hint: First consider the function $F\colon X \to \mathbb{R} \times \mathbb{R}$ given by $F(x) = (f(x), g(x))$.

(c) Suppose X is connected and $f(x) \neq g(x)$ for all $x \in X$. Show that either $f(x) < g(x)$ for all $x \in X$ or $f(x) > g(x)$ for all $x \in X$.

Solutions

Solution. Exercise 1. A *topology* on a set X is a collection $\tau \subseteq \mathcal{P}(X)$ of subsets of X (called *open sets*) such that:

(i) $\varnothing \in \tau$ and $X \in \tau$;

(ii) if $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$;

(iii) if $U, V \in \tau$, then $U \cap V \in \tau$ (equivalently, τ is closed under finite intersections).

A *topological space* is a pair (X, τ) consisting of a set X together with a topology τ on X.

(a) **Three different topologies on $X = \{a, b, c, d\}$.**

(1) The *indiscrete topology*: $\tau_1 = \{\varnothing, X\}$.

(2) The *discrete topology*: $\tau_2 = \mathcal{P}(X)$.

(3) A nontrivial example: $\tau_3 = \{\varnothing, \{a\}, \{a, b\}, X\}$.

(b) **A topological space which is a metric space.** Let $X = \mathbb{R}$ and let $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space, and it induces the usual topology.

(c) **A topological space which is not metrizable.** The indiscrete topology on a set with more than one point is not Hausdorff, hence cannot arise from a metric space (every metric space is Hausdorff). For example, take $Y = \{0, 1\}$ with topology $\{\varnothing, Y\}$.

Solution. Exercise 2. The false statement is (d). For example, $\mathbb{R}$ with the usual topology is Hausdorff but not compact.

Solution. Exercise 3.

(a) (i) A is **open** if $A \in \mathcal{T}$, i.e. A belongs to the topology on X.

(ii) A is **closed** if $X \setminus A$ is open.

(iii) A is **dense** if $\overline{A} = X$.

(b) Since $\overline{A}$ is closed, $X \setminus \overline{A}$ is open. Moreover, $A \subseteq \overline{A}$ implies

$$X \setminus \overline{A} \subseteq X \setminus A.$$

By the universal property of interior, we obtain

$$X \setminus \overline{A} \subseteq \operatorname{Int}(X \setminus A).$$

For the reverse inclusion, let $U \subseteq X$ be open with $U \subseteq X \setminus A$. Then $A \subseteq X \setminus U$. Since $X \setminus U$ is closed and contains A, it contains $\overline{A}$. Thus $U \subseteq X \setminus \overline{A}$. Applying this to $U = \operatorname{Int}(X \setminus A)$ yields

$$\operatorname{Int}(X \setminus A) \subseteq X \setminus \overline{A}.$$

Hence $\operatorname{Int}(X \setminus A) = X \setminus \overline{A}$.

(c) Take $A = (0, 1] \subseteq \mathbb{R}$. Then $\mathrm{Int}(A) = (0, 1)$ and $\overline{A} = [0, 1]$, so A, $\mathrm{Int}(A)$ and $\overline{A}$ are all distinct.

Solution. Exercise 4. The subspace topology on A is

$$\mathcal{T}_A = \{\, U \cap A : U \in \mathcal{T} \,\}.$$

Compute the intersections:

$$\begin{aligned}
\varnothing \cap A &= \varnothing, \\
\{a, b\} \cap A &= \{b\}, \\
\{a, b, c, e, f\} \cap A &= \{b, e\}, \\
\{a, b, d\} \cap A &= \{b, d\}, \\
\{c, d, e, f\} \cap A &= \{d, e\}, \\
\{c, e, f\} \cap A &= \{e\}, \\
\{d\} \cap A &= \{d\}, \\
X \cap A = A &= \{b, d, e\}.
\end{aligned}$$

Hence

$$\mathcal{T}_A = \{\varnothing, \{b\}, \{e\}, \{d\}, \{b, e\}, \{b, d\}, \{d, e\}, A\}.$$

In particular, every singleton in A is open, so $\mathcal{T}_A = \mathcal{P}(A)$ is the discrete topology. Therefore A is Hausdorff.

Solution. Exercise 5. Let $f\colon X \to Y$ and let $\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}$ with the subspace topology. Define $g\colon X \to \Gamma_f$ by $g(x) = (x, f(x))$.

First note that g is bijective: it is injective since $(x, f(x)) = (x', f(x'))$ implies $x = x'$, and it is surjective by definition of Γ_f.

($\Rightarrow$) Assume f is continuous. Consider the map

$$F\colon X \to X \times Y, \qquad F(x) = (x, f(x)).$$

This map is continuous (it is the product of id_X and f). Note that $F = i \circ g$ where $i\colon \Gamma_f \hookrightarrow X \times Y$ is the inclusion. By the universal property of the subspace topology, g is continuous if and only if $i \circ g$ is continuous. Since $i \circ g = F$ is continuous, g is continuous. Moreover, $g^{-1}\colon \Gamma_f \to X$ is given by $g^{-1}(x, y) = x$, i.e. it is the restriction of the projection $\pi_X\colon X \times Y \to X$ to Γ_f, hence continuous. Therefore g is a homeomorphism.

($\Leftarrow$) Assume g is a homeomorphism. Then in particular g is continuous. Let $i\colon \Gamma_f \hookrightarrow X \times Y$ be the inclusion; by definition of the subspace topology i is continuous. Hence $\pi_Y \circ i\colon \Gamma_f \to Y$ is continuous, and

$$f = (\pi_Y \circ i) \circ g.$$

Thus f is a composition of continuous maps, so f is continuous.

Solution. Exercise 6.

(a) Let $K_1, \ldots, K_n \subseteq X$ be compact, and set $K = \bigcup_{i=1}^n K_i$. Let $\mathcal{U}$ be an open cover of K. Then $\mathcal{U}$ restricts to an open cover of each K_i, so for each i there is a finite subcollection $\mathcal{U}_i \subseteq \mathcal{U}$ covering K_i. Then $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is finite and covers K, hence K is compact.

(b) Let X be Hausdorff and let $\{K_i\}_{i\in I}$ be compact subsets of X. Set

$$K := \bigcap_{i\in I} K_i.$$

In a Hausdorff space, compact sets are closed, so each K_i is closed, hence K is closed. Fix $i_0 \in I$. Then K is a closed subset of the compact space K_{i_0}, so K is compact.

Solution. Exercise 7.

(a) We show $\mathrm{cl}_B(A) = B \cap \mathrm{cl}_X(A)$.

First, $B \cap \mathrm{cl}_X(A)$ is closed in B since $\mathrm{cl}_X(A)$ is closed in X, and closed sets in B are of the form $B \cap F$ with F closed in X. Also $A \subseteq B \cap \mathrm{cl}_X(A)$. Thus, by minimality of $\mathrm{cl}_B(A)$,

$$\mathrm{cl}_B(A) \subseteq B \cap \mathrm{cl}_X(A).$$

Conversely, let $C \subseteq B$ be closed in B with $A \subseteq C$. Then $C = B \cap F$ for some closed $F \subseteq X$. Since $A \subseteq F$ and F is closed, $\mathrm{cl}_X(A) \subseteq F$, hence

$$B \cap \mathrm{cl}_X(A) \subseteq B \cap F = C.$$

Therefore $B \cap \mathrm{cl}_X(A) \subseteq \mathrm{cl}_B(A)$, proving equality.

(b) A is dense in B means $\mathrm{cl}_B(A) = B$. Using (a),

$$\mathrm{cl}_B(A) = B \iff B \cap \mathrm{cl}_X(A) = B \iff B \subseteq \mathrm{cl}_X(A).$$

(c) If A is dense in B and B is dense in X, then $B \subseteq \mathrm{cl}_X(A)$, hence

$$X = \mathrm{cl}_X(B) \subseteq \mathrm{cl}_X(\mathrm{cl}_X(A)) = \mathrm{cl}_X(A),$$

so $\mathrm{cl}_X(A) = X$, i.e. A is dense in X.

Solution. Exercise 8.

(a) Let $k := g - f$. Then k is continuous (this can be seen by showing that $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(u, v) \mapsto u - v$ is continuous) and

$$\{x \in X : f(x) < g(x)\} = \{x \in X : k(x) > 0\} = k^{-1}\big((0,\infty)\big),$$

which is open.

(b) Let $F\colon X \to \mathbb{R}\times\mathbb{R}$ be $F(x) = (f(x), g(x))$. Then F is continuous since $F^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ for basic opens $U \times V$. Let $m : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be $m(a, b) = \min\{a, b\}$. Using the subbasis $\{(-\infty, c), (c,\infty)\}$ for $\mathbb{R}$, we have

$$m^{-1}((-\infty, c)) = (-\infty, c) \times \mathbb{R} \ \cup\ \mathbb{R} \times (-\infty, c), \qquad m^{-1}((c,\infty)) = (c,\infty) \times (c,\infty),$$

both open in $\mathbb{R}\times\mathbb{R}$. Hence m is continuous and $h = m \circ F$ is continuous.

Alternative proof: Thanks to Thomas for pointing out the following alternative proof. Let $U = \{x \in X : f(x) \le g(x)\}$ and $V = \{x \in X : g(x) \le f(x)\}$. These are closed: $U = (g - f)^{-1}([0,\infty))$ and $V = (f - g)^{-1}([0,\infty))$ are preimages of a closed set under a continuous function. They cover X since at every point either $f(x) \le g(x)$ or $g(x) \le f(x)$. On U we have $h = f$, and on V we have $h = g$, and these agree on $U \cap V$ since $f(x) = g(x) = \min\{f(x), g(x)\}$ there. By the pasting lemma, h is continuous.

(c) Let $U = \{x \in X : f(x) < g(x)\}$ and $V = \{x \in X : f(x) > g(x)\}$. By (a) both U and V are open. Since $f(x) \neq g(x)$ for all x, we have $U \cap V = \varnothing$ and $U \cup V = X$. By connectedness, either $U = \varnothing$ or $V = \varnothing$, so either $f < g$ everywhere or $f > g$ everywhere.