TMA4190 - Introduction to Topology
Mid semester test

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Exercise 1. Define the notion of a topological space and provide the following examples:

(a) Three different topologies on the set $X = \{a, b, c, d\}$.

(b) A topological space which is a metric space.

(c) A topological space whose topology does not come from a metric space (remember to provide a proof of your claim).

Exercise 2. Which of the following statements is **not** true? Select one alternative, and justify your answer.

☐ The set $\{a, b\}$ equipped with the discrete topology is disconnected.

☐ Every homeomorphism is also an open and closed map.

☐ Every continuous map from a compact space to a Hausdorff space is a closed map.

☐ Every Hausdorff space is compact.

Exercise 3. Let X be a topological space and $A \subseteq X$.

(a) Define what it means for A to be:

(i) open in X,

(ii) closed in X,

(iii) dense in X.

(b) Prove that $\text{Int}(X \setminus A) = X \setminus \overline{A}$ for any subset A of a topological space X.

(c) Give an example of a subset $A \subseteq \mathbb{R}$ (with the usual topology) such that the sets A, $\overline{A}$, and $\text{Int}(A)$ are all different.

Exercise 4. Let $X = \{a, b, c, d, e, f\}$, and let

$$\mathcal{T} = \{\varnothing, \{a, b\}, \{a, b, c, e, f\}, \{a, b, d\}, \{c, d, e, f\}, \{c, e, f\}, \{d\}, X\}$$

be a topology on X. Determine the subspace topology on the subset

$$A = \{b, d, e\}.$$

Is A Hausdorff? Justify your answer.

Exercise 5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a map and define the *graph* of f by

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} \subseteq X \times Y.$$

Equip Γ_f with the subspace topology induced by the product topology on $X \times Y$. Show that f is continuous if and only if the map

$$g : X \to \Gamma_f, \qquad g(x) = (x, f(x)),$$

is a homeomorphism.

Exercise 6. Let X be a space.

(a) Prove that the union of any finite number of compact subsets of X is a compact subset of X.

(b) If X is Hausdorff, prove that the intersection of any collection of compact subsets of X is a compact subset of X.

Exercise 7. Let X be a topological space and let $A \subseteq B \subseteq X$. In this exercise, $\mathrm{cl}_X(-)$ denotes closure taken in X, and $\mathrm{cl}_B(-)$ denotes closure taken in the subspace B (with the subspace topology).

(a) Prove the identity

$$\mathrm{cl}_B(A) = B \cap \mathrm{cl}_X(A).$$

(b) Deduce that A is dense in B (i.e. $\mathrm{cl}_B(A) = B$) if and only if $\mathrm{cl}_X(A) \supseteq B$.

(c) Suppose that A is dense in B and that B is dense in X. Show that A is dense in X.

Exercise 8. Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions (where $\mathbb{R}$ has the usual topology).

(a) Show that the set $\{x \in X : f(x) < g(x)\}$ is open in X.

(b) Define $h : X \to \mathbb{R}$ by $h(x) = \min\{f(x), g(x)\}$. Prove that h is continuous.

Hint: First consider the function $F\colon X \to \mathbb{R} \times \mathbb{R}$ given by $F(x) = (f(x), g(x))$.

(c) Suppose X is connected and $f(x) \neq g(x)$ for all $x \in X$. Show that either $f(x) < g(x)$ for all $x \in X$ or $f(x) > g(x)$ for all $x \in X$.