Lecture 22 - Applications of the fundamental group

Throughout today we identify

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \qquad \text{and} \qquad D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

We will fix $z = 1$ as a convenient basepoint throughout.

1 Change of basepoint and abelian fundamental groups

We begin with the standard change-of-basepoint isomorphism we proved early this term.

Lemma 1. *Let X be a topological space and let $\alpha : I \to X$ be a path from x_0 to x_1. Then the map*

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad \hat{\alpha}([f]) = [\overline{\alpha} * f * \alpha]$$

is an isomorphism.

The next result explains exactly when this isomorphism is independent of the chosen path.

Lemma 2. *Let X be a path-connected space. The following are equivalent:*

1. *For every pair of points $x_0, x_1 \in X$, the change-of-basepoint isomorphism*

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

   is independent of the choice of path $\alpha : x_0 \to x_1$.

2. *$\pi_1(X, x)$ is abelian for one (equivalently every) $x \in X$.*

Proof. Assume first that $\pi_1(X, x_0)$ is abelian. Let $\alpha, \beta : x_0 \to x_1$ be two paths. For $[f] \in \pi_1(X, x_0)$, we have

$$\hat{\alpha}([f]) = [\overline{\alpha} * f * \alpha].$$

Now in $\pi_1(X, x_1)$,

$$[\overline{\alpha} * f * \alpha] = [\overline{\alpha} * \beta]\,[\overline{\beta} * f * \beta]\,[\overline{\beta} * \alpha].$$

Since $\pi_1(X, x_1) \cong \pi_1(X, x_0)$ is abelian, we may commute factors:

$$[\overline{\alpha} * f * \alpha] = [\overline{\beta} * f * \beta]\,[\overline{\alpha} * \beta]\,[\overline{\beta} * \alpha].$$

But

$$[\overline{\alpha} * \beta]\,[\overline{\beta} * \alpha] = [\overline{\alpha} * \alpha] = [e_{x_1}],$$

so

$$\hat{\alpha}([f]) = [\overline{\beta} * f * \beta] = \hat{\beta}([f]).$$

Hence $\hat{\alpha} = \hat{\beta}$, so the change-of-basepoint isomorphism is independent of the chosen path.

Conversely, we prove the contrapositive. Assume that $\pi_1(X, x_0)$ is not abelian. Then there exist $[a], [b] \in \pi_1(X, x_0)$ such that

$$[a][b] \neq [b][a].$$

Equivalently,

$$[b]^{-1}[a][b] \neq [a].$$

Let c_{x_0} denote the constant path at x_0. Then

$$\widehat{c_{x_0}}([a]) = [a].$$

On the other hand, viewing b as a path from x_0 to x_0, we have

$$\hat{b}([a]) = [\overline{b} * a * b] = [b]^{-1}[a][b] \neq [a].$$

Thus

$$\hat{b} \neq \widehat{c_{x_0}}.$$

So the change-of-basepoint isomorphism depends on the chosen path.

This proves the contrapositive, and hence if the change-of-basepoint isomorphism is independent of path, then $\pi_1(X, x_0)$ is abelian. □

Example 3. Since $\pi_1(S^1, 1) \cong \mathbb{Z}$ is abelian, the change-of-basepoint isomorphism on S^1 is independent of the chosen path.

2 Degree of maps $S^1 \to S^1$

Fix once and for all a generator

$$\iota \in \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Definition 4. Let $f : S^1 \to S^1$ be continuous. Choose any path a in S^1 from $f(1)$ to 1. Since $\pi_1(S^1, 1)$ is abelian, Theorem 2 shows that the change-of-basepoint isomorphism

$$\hat{a} : \pi_1(S^1, f(1)) \to \pi_1(S^1, 1)$$

is independent of the choice of a. We define the *degree* of f, denoted $\deg(f) \in \mathbb{Z}$, by the condition

$$(\hat{a} \circ f_*)(\iota) = \deg(f)\, \iota.$$

Intuitively, $\deg(f)$ counts how many times the image of S^1 winds around the circle under f, with sign.

Proposition 5. *Let $f, g : S^1 \to S^1$ be continuous. Then:*

(1) *If $f \simeq g$, then $\deg(f) = \deg(g)$.*

(2) *If $c : S^1 \to S^1$ is constant, then $\deg(c) = 0$.*

(3) *If $f_n : S^1 \to S^1$ is given by $f_n(z) = z^n$ for any $n \in \mathbb{Z}$, then $\deg(f_n) = n$.*

(4) $\deg(g \circ f) = \deg(g)\deg(f)$.

Proof. Part (1) follows from homotopy invariance of the induced map on fundamental groups.

Part (2) is immediate, since the induced map of a constant map is trivial.

For (3), note that $f_n(1) = 1$, so f preserves the basepoint and no change-of-basepoint adjustment is needed. Recall that $\pi_1(S^1, 1) \cong \mathbb{Z}$ via the lifting correspondence

$$\Phi : \pi_1(S^1, 1) \to \mathbb{Z},$$

where $\Phi([\gamma])$ is the endpoint of the lift of γ to $\mathbb{R}$ starting at 0, under the covering map

$$p : \mathbb{R} \to S^1, \qquad p(t) = e^{2\pi i t}.$$

Let $\gamma(s)\colon I \to S^1$ be the loop in S^1 based at 1 given by $\gamma(s) = e^{2\pi i s}$. Then $[\gamma]$ is a generator of $\pi_1(S^1, 1)$. Then

$$(f_n \circ \gamma)(s) = \gamma(s)^n = e^{2\pi i n s}.$$

This lifts to the path $\tilde{\gamma}(s) = ns$ in $\mathbb{R}$, whose endpoint is n. Hence

$$\Phi([f_n \circ \gamma]) = n.$$

Since $\Phi([\gamma]) = 1$, it follows that f_{n*} sends the generator to n times the generator. Therefore

$$\deg(f_n) = n.$$

For (4), the induced map of a composite is the composite of the induced maps, and multiplication by $\deg(f)$ followed by multiplication by $\deg(g)$ is multiplication by $\deg(g)\deg(f)$. □

Example 6. The identity map has degree 1, the constant map has degree 0, and the map

$$r : S^1 \to S^1, \qquad r(z) = \overline{z}$$

has degree -1. Indeed, on S^1 we have $\overline{z} = z^{-1}$ so $r = f_{-1}$ and part (3) applies directly.

Remark 7. One can show that two maps $f, g : S^1 \to S^1$ are homotopic if and only if they have the same degree. Thus homotopy classes of maps $S^1 \to S^1$ are classified by degree.

3 Brouwer's fixed point theorem

Proposition 8. *Let $f : S^1 \to S^1$ be continuous. If there exists a continuous map $F : D^2 \to S^1$ such that*

$$F|_{S^1} = f,$$

then $\deg(f) = 0$.

Proof. Consider the inclusion $i : S^1 \hookrightarrow D^2$. Since D^2 is contractible, its fundamental group is trivial, so

$$i_* : \pi_1(S^1, 1) \to \pi_1(D^2, 1)$$

is the zero homomorphism. Since $f = F \circ i$, we have

$$f_* = F_* \circ i_*.$$

Therefore f_* is the zero homomorphism. Since f_* is trivial, $\hat{a} \circ f_*$ is also trivial, so

$$(\hat{a} \circ f_*)(\iota) = 0.$$

By definition of degree, this implies $\deg(f) = 0$. □

Corollary 9. *There is no retraction $r : D^2 \to S^1$.*

Proof. Suppose $r : D^2 \to S^1$ were a retraction. Then

$$r|_{S^1} = \mathrm{id}_{S^1}.$$

But id_{S^1} has degree 1, while by Theorem 8, any map $S^1 \to S^1$ that extends over D^2 must have degree 0. This is a contradiction. □

Remark 10. Degree is not strictly necessary here. All we use is the following: if

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

is the identity, then so is the induced map

$$\mathbb{Z} \to 0 \to \mathbb{Z}$$

on fundamental groups, but this is the zero map, a contradiction.

We now derive Brouwer's fixed point theorem for the disk.

Theorem 11 (Brouwer fixed point theorem)**.** *Every continuous map*

$$F : D^2 \to D^2$$

has a fixed point.

Proof. Assume, for contradiction, that F has no fixed point. Then for each $x \in D^2$, the points $F(x)$ and x are distinct. Draw the ray starting at $F(x)$ and passing through x. This ray meets the boundary circle S^1 in a unique point. Define

$$r(x) = \text{the unique point where the ray from } F(x) \text{ through } x \text{ meets } S^1.$$

Then $r : D^2 \to S^1$ is continuous, and if $x \in S^1$, the ray from $F(x)$ through x meets S^1 at x itself, so

$$r(x) = x \qquad \text{for all } x \in S^1.$$

Thus r is a retraction $D^2 \to S^1$, contradicting Theorem 9. □

4 The Fundamental Theorem of Algebra

We now turn to the second major application.

Theorem 12 (Fundamental Theorem of Algebra)**.** *Let*

$$f(z) = z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n$$

be a polynomial with complex coefficients, where $n > 0$. Then there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

Consequently, f has exactly n complex roots counted with multiplicity.

Proof. Once existence of one root is known, the statement about n roots counted with multiplicity follows by induction on the degree: if c is a root, then $f(z) = (z - c)q(z)$ for some polynomial q of degree $n - 1$.

So it is enough to prove that f has at least one complex root.

We may as well assume that

$$f(x) \neq 0 \qquad \text{for all } x \in S^1.$$

Indeed, if $f(x) = 0$ for some $x \in S^1$, then we are done.

Under this assumption, we may define

$$\widehat{f} : S^1 \to S^1, \qquad \widehat{f}(x) = \frac{f(x)}{|f(x)|}.$$

We shall compute $\deg(\widehat{f})$ in two ways.

First computation. Suppose first that

$$f(x) \neq 0 \qquad \text{for all } x \in \mathbb{C} \text{ with } |x| \leq 1.$$

Let $x \in S^1$ and $t \in I = [0,1]$. Then $|x| = 1$, so

$$|tx| = t|x| = t \leq 1.$$

Hence $tx \in D^2$, and by assumption $f(tx) \neq 0$. Therefore we may define

$$h : S^1 \times I \to S^1, \qquad h(x,t) = \frac{f(tx)}{|f(tx)|}.$$

This is well-defined because $tx \in D^2$ for all $x \in S^1$ and $t \in I$, and by assumption $f(tx) \neq 0$.

Now

$$h(x,0) = \frac{f(0)}{|f(0)|},$$

which is constant in x, while

$$h(x,1) = \frac{f(x)}{|f(x)|} = \widehat{f}(x).$$

Thus h is a homotopy from a constant map to $\widehat{f}$. By homotopy invariance of degree,

$$\deg(\widehat{f}) = 0.$$

Second computation. Suppose next that

$$f(x) \neq 0 \qquad \text{for all } x \in \mathbb{C} \text{ with } |x| \geq 1.$$

Define

$$k : S^1 \times I \to \mathbb{C}$$

by

$$k(x,t) = x^n + tc_1x^{n-1} + t^2c_2x^{n-2} + \cdots + t^nc_n.$$

For $t > 0$, this can be rewritten as

$$k(x,t) = t^n f(x/t).$$

Indeed,

$$t^n f(x/t) = t^n \left((x/t)^n + c_1(x/t)^{n-1} + \cdots + c_n \right) = x^n + tc_1x^{n-1} + \cdots + t^nc_n.$$

We claim that $k(x,t) \neq 0$ for all $(x,t) \in S^1 \times I$. If $t = 0$, then

$$k(x,0) = x^n \neq 0$$

since $x \in S^1$. If $t > 0$, then

$$|k(x,t)| = t^n \, |f(x/t)|.$$

Now $|x| = 1$ and $0 < t \leq 1$, so

$$|x/t| = \frac{1}{t} \geq 1.$$

By the assumption of this case, $f(x/t) \neq 0$, hence $|f(x/t)| > 0$. Since $t^n > 0$, it follows that $|k(x,t)| > 0$, so $k(x,t) \neq 0$.

Therefore we may define

$$j : S^1 \times I \to S^1, \qquad j(x,t) = \frac{k(x,t)}{|k(x,t)|}.$$

Now

$$j(x,0) = \frac{x^n}{|x^n|} = x^n$$

and

$$j(x,1) = \frac{x^n + c_1 x^{n-1} + \cdots + c_n}{|x^n + c_1 x^{n-1} + \cdots + c_n|} = \widehat{f}(x).$$

Thus j is a homotopy from the map

$$f_n(x) = x^n$$

to $\widehat{f}$. By homotopy invariance and Theorem 5,

$$\deg(\widehat{f}) = \deg(f_n) = n.$$

We have shown:

$$\text{if } f \text{ has no zeros in } D^2, \text{ then } \deg(\widehat{f}) = 0,$$

and

$$\text{if } f \text{ has no zeros outside } D^2, \text{ then } \deg(\widehat{f}) = n.$$

Suppose for contradiction that f has no complex root. Then both assumptions hold simultaneously, and $\deg(\widehat{f})$ is both 0 and n, a contradiction. Hence f has at least one complex root. □