Topology Problems

1. Let $\rho\colon X \times X \to \mathbb{R}_{\geq 0}$ be a metric on X. Prove that the function

$$d(x, y) = \min\{\rho(x, y), 1\}$$

is also a metric on X.

2. A function $f\colon X \to \mathbb{R}$ is said to be *locally bounded* if for each $a \in X$ there exist an open neighbourhood U of a and a number $M > 0$ such that $|f(x)| \leq M$ for all $x \in U$.

Prove that if X is compact and $f\colon X \to \mathbb{R}$ is locally bounded, then f is bounded.

3. Prove that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is homeomorphic to $S^1 \times \mathbb{R}$.

4. Let X be a topological space and let $K \subseteq X$ be a compact subspace such that $K \neq X$. Assume that for every pair of points $x, y \in X$ with $x \neq y$ there exists a continuous map

$$f\colon X \to [0, 1]$$

such that $f(x) = 0$ and $f(y) = 1$.

(i) Show that X is Hausdorff.

(ii) Let $x \notin K$. Show that there exists a continuous map

$$f\colon X \to [0, 1]$$

such that $f(x) = 1$ and $f(K) \subseteq [0, 1/2)$.

5. Let

$$A = \left\{ \frac{1}{n^2} \;\middle|\; n \in \mathbb{Z}_{>0} \right\} \subseteq \mathbb{R}.$$

What is the closure of A in $\mathbb{R}$ with the standard topology?

6. Let (X, τ) be a topological space. Define an equivalence relation $\sim$ on X by declaring $x \sim y$ if and only if for every surjective continuous map $f\colon X \to Y$ into a Hausdorff space Y, we have $f(x) = f(y)$.

Let

$$HX := X/\!\sim$$

with the quotient topology. Prove that HX is Hausdorff.

7. Show that if X is Hausdorff and $A \subseteq X$ is a retract of X, then A is closed in X.