Topology Exercises: Generating Topologies

Exercises and Solutions

Exercise 1 (4.1)**.** Let $X = \{a, b, c, d, e\}$, and let

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, d, e\}, \{b\}, \{b, d, e\}, \{b, c, d, e\}, \{c, d, e\}, \{d, e\}, X\}$$

be a topology on X. Show that $\mathcal{S} = \{\{a, b\}, \{b, d, e\}, \{c, d, e\}\}$ is a subbasis for $\mathcal{T}$. Is $\mathcal{S}' = \{\{a, b\}, \{b, c, d, e\}, \{d, e\}\}$ a subbasis for $\mathcal{T}$?

Solution. Recall that a subbasis $\mathcal{S}$ for a topology $\mathcal{T}$ is a collection of subsets such that $\mathcal{T}$ is the collection of all unions of finite intersections of elements from $\mathcal{S}$.

Let's find the basis $\mathcal{B}$ generated by $\mathcal{S}$ (all finite intersections of elements from $\mathcal{S}$):

$$\begin{aligned}
&\{a, b\} \cap \{b, d, e\} = \{b\} \\
&\{a, b\} \cap \{c, d, e\} = \emptyset \\
&\{b, d, e\} \cap \{c, d, e\} = \{d, e\} \\
&\{a, b\} \cap \{b, d, e\} \cap \{c, d, e\} = \{b\} \cap \{c, d, e\} = \emptyset
\end{aligned}$$

So the basis is $\mathcal{B} = \{\{b\}, \{d, e\}, \{a, b\}, \{b, d, e\}, \{c, d, e\}\}$ (including the sets themselves and their intersections).

Now, let's generate all open sets from $\mathcal{B}$ (all unions of basis elements):

- Single sets: $\{b\}, \{d, e\}, \{a, b\}, \{b, d, e\}, \{c, d, e\}$
- Unions:
  - $\{b\} \cup \{d, e\} = \{b, d, e\}$ (already listed)
  - $\{b\} \cup \{a, b\} = \{a, b\}$ (already listed)
  - $\{b\} \cup \{b, d, e\} = \{b, d, e\}$ (already listed)
  - $\{b\} \cup \{c, d, e\} = \{b, c, d, e\}$
  - $\{d, e\} \cup \{a, b\} = \{a, b, d, e\}$
  - $\{d, e\} \cup \{b, d, e\} = \{b, d, e\}$ (already listed)
  - $\{d, e\} \cup \{c, d, e\} = \{c, d, e\}$ (already listed)
  - $\{a, b\} \cup \{b, d, e\} = \{a, b, d, e\}$ (already listed)
  - $\{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\} = X$

- $\{b, d, e\} \cup \{c, d, e\} = \{b, c, d, e\}$ (already listed)

- Larger unions:
  - $\{b\} \cup \{d, e\} \cup \{a, b\} = \{a, b, d, e\}$
  - $\{b\} \cup \{d, e\} \cup \{c, d, e\} = \{b, c, d, e\}$
  - All three: $\{a, b\} \cup \{b, d, e\} \cup \{c, d, e\} = X$
  - And the empty set is included by taking the empty union

So we get: $\{\emptyset, \{b\}, \{d, e\}, \{a, b\}, \{b, d, e\}, \{c, d, e\}, \{b, c, d, e\}, \{a, b, d, e\}, X\}$, which is exactly $\mathcal{T}$. Therefore, $\mathcal{S}$ is a subbasis for $\mathcal{T}$.

For $\mathcal{S}' = \{\{a, b\}, \{b, c, d, e\}, \{d, e\}\}$:

- Finite intersections:

$$\{a, b\} \cap \{b, c, d, e\} = \{b\}$$
$$\{a, b\} \cap \{d, e\} = \emptyset$$
$$\{b, c, d, e\} \cap \{d, e\} = \{d, e\}$$
$$\text{All three: } \{a, b\} \cap \{b, c, d, e\} \cap \{d, e\} = \{b\} \cap \{d, e\} = \emptyset$$

- So the basis is $\{\{b\}, \{d, e\}, \{a, b\}, \{b, c, d, e\}\}$.

- From this, we cannot obtain $\{c, d, e\}$, which is in $\mathcal{T}$. The smallest set containing c we can generate is $\{b, c, d, e\}$ (or X), but this contains b, whereas $\{c, d, e\}$ does not. Specifically, any open set containing c must contain b, so $\{c, d, e\}$ cannot be formed.

Therefore, $\mathcal{S}'$ is **not** a subbasis for $\mathcal{T}$.

Exercise 2 (4.2)**.** Let $\mathcal{B}$ be the collection of all open intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$ denoted by $\mathcal{T}_{\text{std}}$.

(b) Show that $\mathcal{T}_{\text{std}} = \mathcal{T}_d$ where $\mathcal{T}_d$ is the metric topology obtained from the metric $d(x, y) = |x - y|$.

Solution. (a) To show $\mathcal{B}$ is a basis, we need to verify two properties:

(i) For every $x \in \mathbb{R}$, there exists $B \in \mathcal{B}$ such that $x \in B$. Indeed, for any $x \in \mathbb{R}$, take $(x - 1, x + 1) \in \mathcal{B}$, then $x \in (x - 1, x + 1)$.

(ii) For any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. If $x \in B_1 \cap B_2$, then $x \in (a_1, b_1)$ and $x \in (a_2, b_2)$. Let $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$. Then $a < x < b$, so $x \in (a, b)$. Also, $(a, b) \subseteq (a_1, b_1) \cap (a_2, b_2)$. So we can take $B_3 = (a, b)$.

Therefore, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

(b) Recall that in a metric space (X, d), the metric topology $\mathcal{T}_d$ has as basis the collection of all open balls $B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$.

For $\mathbb{R}$ with $d(x, y) = |x-y|$, we have $B_d(x, \epsilon) = (x-\epsilon, x+\epsilon)$, which is an open interval. So every open ball is in $\mathcal{B}$.

Conversely, for any $(a, b) \in \mathcal{B}$ and $x \in (a, b)$, let $\epsilon = \min(x - a, b - x) > 0$. Then $B_d(x, \epsilon) = (x - \epsilon, x + \epsilon) \subseteq (a, b)$. So (a, b) is open in the metric topology.

Therefore, $\mathcal{T}_{\text{std}} = \mathcal{T}_d$.

Exercise 3 (4.3)**.** Show that

$$\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$$

is a subbasis for the standard topology on $\mathbb{R}$.

Solution. We need to show that the topology generated by $\mathcal{S}$ equals $\mathcal{T}_{\text{std}}$.

The basis generated by $\mathcal{S}$ consists of all finite intersections of elements from $\mathcal{S}$. Consider intersections of the form:

$$(a, \infty) \cap (-\infty, b) = (a, b) \quad \text{if } a < b$$

If $a \geq b$, the intersection is empty. Also, intersections of multiple sets of the same type give either the same type or empty set:

$$(a, \infty) \cap (c, \infty) = (\max(a, c), \infty)$$

$$(-\infty, b) \cap (-\infty, d) = (-\infty, \min(b, d))$$

So the basis $\mathcal{B}$ generated by $\mathcal{S}$ consists of all open intervals (a, b), all rays (a, ∞) and $(-\infty, b)$, and the empty set. But note that any ray (a, ∞) can be written as $\bigcup_{n=1}^{\infty}(a, a+n)$, a union of open intervals. Similarly, $(-\infty, b) = \bigcup_{n=1}^{\infty}(b-n, b)$.

Therefore, the topology generated by $\mathcal{B}$ (which is the same as the topology generated by $\mathcal{S}$) contains all open intervals, and thus contains $\mathcal{T}_{\text{std}}$. Conversely, since every element of $\mathcal{S}$ is open in the standard topology (as it can be written as a union of open intervals), the topology generated by $\mathcal{S}$ is contained in $\mathcal{T}_{\text{std}}$.

Hence, $\mathcal{S}$ is a subbasis for $\mathcal{T}_{\text{std}}$.

Exercise 4 (4.4)**.** Let $\mathbb{Q}$ denote the set of rational numbers, and let $\mathbb{R}$ denote the set of real numbers. Show that

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}$ where $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

Solution. We need to show two things:

(i) $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$.

(ii) The topology generated by $\mathcal{B}$ equals $\mathcal{T}_{\text{std}}$.

For (i):

1. For any $x \in \mathbb{R}$, there exist rational numbers a, b such that $a < x < b$ (by density of $\mathbb{Q}$ in $\mathbb{R}$). So $x \in (a, b) \in \mathcal{B}$.

2. For $(a_1, b_1), (a_2, b_2) \in \mathcal{B}$ and $x \in (a_1, b_1) \cap (a_2, b_2)$, let $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$. Since a_1, a_2, b_1, b_2 are rational, a and b are rational. Also, $a < x < b$, so $x \in (a, b) \subseteq (a_1, b_1) \cap (a_2, b_2)$.

Thus $\mathcal{B}$ is a basis.

For (ii): Let $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. Since every element of $\mathcal{B}$ is an open interval, $\mathcal{B} \subseteq \mathcal{T}_{\text{std}}$, so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}_{\text{std}}$.

Conversely, let $U \in \mathcal{T}_{\text{std}}$ and $x \in U$. Since $\mathcal{T}_{\text{std}}$ is generated by all open intervals, there exists an open interval (c, d) with $c, d \in \mathbb{R}$ such that $x \in (c, d) \subseteq U$. By density of $\mathbb{Q}$, choose $a, b \in \mathbb{Q}$ such that $c < a < x < b < d$. Then $x \in (a, b) \subseteq (c, d) \subseteq U$, and $(a, b) \in \mathcal{B}$. So U is a union of basis elements from $\mathcal{B}$, hence $U \in \mathcal{T}_{\mathcal{B}}$.

Therefore, $\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\text{std}}$.

Exercise 5 (4.5)**.** Let $\mathcal{B}$ be the collection of all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the lower limit topology on $\mathbb{R}$.

(b) Find the closure of the subset $(0, 1)$ of $\mathbb{R}$ given the lower limit topology.

Solution. (a) To show $\mathcal{B}$ is a basis:

(i) For any $x \in \mathbb{R}$, $x \in [x, x + 1) \in \mathcal{B}$.

(ii) For $[a_1, b_1), [a_2, b_2) \in \mathcal{B}$ and $x \in [a_1, b_1) \cap [a_2, b_2)$, let $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$. Since x is in both intervals, we have $a_1 \leq x < b_1$ and $a_2 \leq x < b_2$, so $a \leq x < b$. Thus $x \in [a, b) \subseteq [a_1, b_1) \cap [a_2, b_2)$.

Therefore, $\mathcal{B}$ is a basis. The topology generated is called the lower limit topology (or Sorgenfrey topology).

(b) Let $\mathcal{T}_{ll}$ denote the lower limit topology. We want to find $\overline{(0,1)}$ in $(\mathbb{R}, \mathcal{T}_{ll})$. Certainly $(0, 1) \subseteq \overline{(0, 1)}$.

Recall that $x \in \overline{A}$ if and only if every open neighborhood of x intersects A.

Consider $x \in \mathbb{R}$:

- If $x < 0$: The set $[x, 0)$ is open and $[x, 0) \cap (0, 1) = \emptyset$. So $x \notin \overline{(0, 1)}$.
- If $x = 0$: For any basic open set $[a, b)$ containing 0, we have $a \leq 0 < b$. Then $[a, b) \cap (0, 1) \neq \emptyset$ because it contains the open interval $(0, \min(b, 1))$. So $0 \in \overline{(0, 1)}$.
- If $x = 1$: Consider the basic open set $[1, 2)$. Then $[1, 2) \cap (0, 1) = \emptyset$. So $1 \notin \overline{(0, 1)}$.
- If $x > 1$: The set $[x, x + 1)$ is open and $[x, x + 1) \cap (0, 1) = \emptyset$. So $x \notin \overline{(0, 1)}$.

Therefore, $\overline{(0,1)} = [0,1)$ in the lower limit topology.

Exercise 6 (4.6)**.** For each $n \in \mathbb{Z}$, let

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

Show that the collection $\mathcal{B} = \{B(n) \mid n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$. The topology generated by $\mathcal{B}$ is known as the digital line topology on $\mathbb{Z}$.

Solution. To prove that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$, we must verify two conditions:

1. For every $x \in \mathbb{Z}$, there is at least one basis element $B \in \mathcal{B}$ containing x.
2. If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Condition 1 (Covering): Let $x \in \mathbb{Z}$. Consider the basis element $B(x)$.

- If x is odd, $B(x) = \{x\}$, which clearly contains x.
- If x is even, $B(x) = \{x-1, x, x+1\}$, which also contains x.

In either case, $x \in B(x)$. Thus, $\bigcup_{B \in \mathcal{B}} B = \mathbb{Z}$.

Condition 2 (Intersection): Let $B(n), B(m) \in \mathcal{B}$ and suppose $x \in B(n) \cap B(m)$. We need to find a basis element B_3 such that $x \in B_3 \subseteq B(n) \cap B(m)$. We consider the parity of x.

Case 1: x **is odd.** By definition, if x is odd, the basis element generated by x is $B(x) = \{x\}$. Since $x \in B(n) \cap B(m)$, the singleton set $\{x\}$ is a subset of the intersection. Thus:

$$x \in B(x) \subseteq B(n) \cap B(m).$$

We choose $B_3 = B(x)$.

Case 2: x **is even.** First, we observe which basis elements contain even numbers.

- If k is odd, $B(k) = \{k\}$. This set contains no even numbers.
- If k is even, $B(k) = \{k-1, k, k+1\}$. The elements $k-1$ and $k+1$ are odd. The only even element in $B(k)$ is k itself.

Since x is even and $x \in B(n)$, n must be even and $n = x$. Similarly, since $x \in B(m)$, m must be even and $m = x$. Therefore, $B(n) = B(x)$ and $B(m) = B(x)$. The intersection is simply:

$$B(n) \cap B(m) = B(x).$$

We choose $B_3 = B(x)$, which trivially satisfies $x \in B_3 \subseteq B(n) \cap B(m)$.

Since both conditions are satisfied, $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

Exercise 7 (4.7)**.** Let $\mathcal{B}$ be the collection of all subsets of the form $A_{a,b} = \{az + b \mid z \in \mathbb{Z}\}$ of $\mathbb{Z}$, where $a, b \in \mathbb{Z}$ and $a \neq 0$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

(b) Show that there are infinitely many primes by using the topology generated by $\mathcal{B}$.

Solution. (a) We verify the basis axioms.

(i) Covering. For any $x \in \mathbb{Z}$, the set $A_{1,x} = \{z + x \mid z \in \mathbb{Z}\} = \mathbb{Z}$ is in $\mathcal{B}$ and contains x. Hence $\bigcup_{B \in \mathcal{B}} B = \mathbb{Z}$.

(ii) Intersection. Let $A_{a,b}, A_{c,d} \in \mathcal{B}$ and suppose $x \in A_{a,b} \cap A_{c,d}$. Write $x = az_0 + b$ and $x = cw_0 + d$ for some $z_0, w_0 \in \mathbb{Z}$. Then

$$A_{a,x} = \{az + x \mid z \in \mathbb{Z}\} = \{az + (az_0 + b) \mid z \in \mathbb{Z}\} = \{a(z + z_0) + b \mid z \in \mathbb{Z}\} = A_{a,b},$$

and similarly $A_{c,x} = A_{c,d}$.

Let $m = \operatorname{lcm}(|a|, |c|)$ (a positive integer). Since m is divisible by $|a|$ and $|c|$, we have $m = |a|u = |c|v$ for some $u, v \in \mathbb{Z}$. Then for any $z \in \mathbb{Z}$,

$$mz + x = (|a|u)z + x = a(\pm uz) + x \in A_{a,x},$$

and similarly $mz + x \in A_{c,x}$. Hence

$$x \in A_{m,x} \subseteq A_{a,x} \cap A_{c,x} = A_{a,b} \cap A_{c,d}.$$

Thus the basis intersection axiom holds, so $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

(b) Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. First note that each arithmetic progression $A_{a,b}$ is *clopen* (both open and closed). Indeed, $A_{a,b}$ is open by definition, and writing $n = |a| > 0$, its complement is the union of the other congruence classes modulo n:

$$\mathbb{Z} \setminus A_{a,b} = \bigcup_{r=1}^{n-1} A_{n,b+r},$$

which is a union of open sets, hence open. Therefore $A_{a,b}$ is closed.

Now suppose (for contradiction) that there are only finitely many primes $p_1, \ldots, p_k$. Consider

$$S = \bigcup_{i=1}^{k} A_{p_i,0},$$

the set of integers divisible by at least one prime in this list. Each $A_{p_i,0}$ is closed, so S is a finite union of closed sets and hence closed. Every integer $n \in \mathbb{Z} \setminus \{-1, 1\}$ has at least one prime divisor, so $n \in S$. Thus

$$\mathbb{Z} \setminus S = \{-1, 1\}.$$

Since S is closed, its complement $\{-1, 1\}$ is open.

But any nonempty open set in this topology contains a basis element: if U is nonempty and open, then U is a union of basis elements, so it contains some $A_{a,b}$. Every $A_{a,b}$ is infinite, so no finite set (in particular $\{-1, 1\}$) can be open. This contradiction shows that the primes cannot be finite in number.

Hence there are infinitely many primes.

Exercise 8 (4.8)**.** Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Show that a subset A of X is dense in X if and only if every non-empty basis element in $\mathcal{B}$ intersects A.

Solution. Recall: A is dense in X if $\overline{A} = X$, which is equivalent to: for every non-empty open set $U \subseteq X$, $U \cap A \neq \emptyset$.

($\Rightarrow$) Suppose A is dense. Let $B \in \mathcal{B}$ be non-empty. Then B is open, so by the characterization of dense sets, $B \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every non-empty basis element intersects A. Let U be any non-empty open set in X. Since $\mathcal{B}$ is a basis, there exists some $B \in \mathcal{B}$ such that $B \subseteq U$ and $B \neq \emptyset$. By assumption, $B \cap A \neq \emptyset$. But then $U \cap A \supseteq B \cap A \neq \emptyset$. Therefore, every non-empty open set intersects A, so A is dense.