Solutions to Exercises 7.1–7.8

Homotopy type and the fundamental group

Exercise 7.1. Let X, Y and Z be topological spaces. Show that if $f\colon X \to Y$ is homotopic to $f'\colon X \to Y$ and $g\colon Y \to Z$ is homotopic to $g'\colon Y \to Z$, then $g \circ f\colon X \to Z$ is homotopic to $g' \circ f'\colon X \to Z$.

Solution. Let

$$H\colon X \times I \to Y$$

be a homotopy from f to f', and let

$$K\colon Y \times I \to Z$$

be a homotopy from g to g'.

First define

$$L\colon X \times I \to Z, \qquad L(x,t) = g(H(x,t)).$$

Since H and g are continuous, L is continuous. Moreover,

$$L(x,0) = g(H(x,0)) = g(f(x)), \qquad L(x,1) = g(H(x,1)) = g(f'(x)).$$

Thus $g \circ f \simeq g \circ f'$.

Next define

$$M\colon X \times I \to Z, \qquad M(x,t) = K(f'(x),t).$$

This is continuous, and

$$M(x,0) = K(f'(x),0) = g(f'(x)), \qquad M(x,1) = K(f'(x),1) = g'(f'(x)).$$

Thus $g \circ f' \simeq g' \circ f'$.

By transitivity of homotopy, it follows that

$$g \circ f \simeq g' \circ f'.$$

Exercise 7.2. Let X be a topological space. Show that X is path-connected if and only if every two constant maps $c_1\colon X \to X$ and $c_2\colon X \to X$ are homotopic.

Solution. Suppose first that X is path-connected. Let $c_1, c_2\colon X \to X$ be constant maps, say

$$c_1(x) = x_0, \qquad c_2(x) = x_1$$

for all $x \in X$. Since X is path-connected, there exists a path

$$\alpha\colon I \to X$$

from x_0 to x_1. Define

$$H\colon X \times I \to X, \qquad H(x,t) = \alpha(t).$$

This is continuous, and

$$H(x,0) = \alpha(0) = x_0 = c_1(x), \qquad H(x,1) = \alpha(1) = x_1 = c_2(x).$$

Hence $c_1 \simeq c_2$.

Conversely, suppose every two constant maps $X \to X$ are homotopic. Let $x_0, x_1 \in X$. Define constant maps

$$c_0(x) = x_0, \qquad c_1(x) = x_1.$$

By assumption there exists a homotopy

$$H\colon X \times I \to X$$

from c_0 to c_1. Fix any point $x \in X$. Then the map

$$\alpha\colon I \to X, \qquad \alpha(t) = H(x,t)$$

is a path from x_0 to x_1, since

$$\alpha(0) = H(x,0) = x_0, \qquad \alpha(1) = H(x,1) = x_1.$$

Thus any two points of X can be joined by a path, so X is path-connected.

Exercise 7.3. Let X be a topological space, and let x_0, x_1, x_2 be points in X. Show that if $\alpha\colon I \to X$ is a path from x_0 to x_1, $\beta\colon I \to X$ is a path from x_1 to x_2 and $\gamma = \alpha * \beta$, then

$$\widehat{\gamma} = \widehat{\beta} \circ \widehat{\alpha}.$$

Solution. Recall that for a path η from a to b, the change-of-basepoint isomorphism

$$\widehat{\eta}\colon \pi_1(X,a) \to \pi_1(X,b)$$

is defined by

$$\widehat{\eta}([f]) = [\overline{\eta} * f * \eta].$$

Let $[f] \in \pi_1(X, x_0)$. Then

$$\widehat{\gamma}([f]) = [\overline{\gamma} * f * \gamma].$$

Since $\gamma = \alpha * \beta$, we have

$$\overline{\gamma} = \overline{\beta} * \overline{\alpha}.$$

Hence

$$\widehat{\gamma}([f]) = [(\overline{\beta} * \overline{\alpha}) * f * (\alpha * \beta)].$$

By associativity of path concatenation up to path homotopy, this is equal to

$$[\overline{\beta} * (\overline{\alpha} * f * \alpha) * \beta].$$

But

$$\widehat{\alpha}([f]) = [\overline{\alpha} * f * \alpha].$$

Therefore

$$\widehat{\beta}(\widehat{\alpha}([f])) = \widehat{\beta}([\overline{\alpha} * f * \alpha]) = [\overline{\beta} * (\overline{\alpha} * f * \alpha) * \beta].$$

Thus

$$\widehat{\gamma}([f]) = (\widehat{\beta} \circ \widehat{\alpha})([f])$$

for every $[f] \in \pi_1(X, x_0)$, and so

$$\widehat{\gamma} = \widehat{\beta} \circ \widehat{\alpha}.$$

Exercise 7.4. Let S^2 be the 2-sphere considered as a subspace of $\mathbb{R}^3$ with the standard topology. Show that

$$\pi_1(S^2 \setminus \{(0,0,1)\}, s)$$

is the trivial group, where $s = (0,0,-1)$.

Solution. The space $S^2 \setminus \{(0,0,1)\}$ is homeomorphic to $\mathbb{R}^2$ by stereographic projection from the north pole $(0,0,1)$. Since homeomorphic spaces have isomorphic fundamental groups, it follows that

$$\pi_1(S^2 \setminus \{(0,0,1)\}, s) \cong \pi_1(\mathbb{R}^2, 0).$$

But $\mathbb{R}^2$ is contractible, hence simply connected, so

$$\pi_1(\mathbb{R}^2, 0) = 0.$$

Therefore

$$\pi_1(S^2 \setminus \{(0,0,1)\}, s) = 0.$$

Exercise 7.5. Let X be a Hausdorff space, and let A be a subspace of X. Show that if A is a retract of X, then A is a closed subset of X.

Solution. Let

$$r \colon X \to A$$

be a retraction, so $r(a) = a$ for all $a \in A$.

Consider the map

$$F \colon X \to X \times X, \qquad F(x) = (x, r(x)).$$

This map is continuous.

Now let

$$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$$

be the diagonal. Since X is Hausdorff, Δ_X is closed in $X \times X$.

We claim that

$$A = F^{-1}(\Delta_X).$$

Indeed,

$$x \in F^{-1}(\Delta_X) \iff (x, r(x)) \in \Delta_X \iff x = r(x).$$

If $x \in A$, then $r(x) = x$, so $x \in F^{-1}(\Delta_X)$. Conversely, if $x = r(x)$, then $r(x) \in A$, hence $x \in A$. Thus $A = F^{-1}(\Delta_X)$.

Since F is continuous and Δ_X is closed, it follows that A is closed in X.

Exercise 7.6. Show that the relation of homotopy equivalence is an equivalence relation on any set of topological spaces.

Solution. We must show reflexivity, symmetry, and transitivity.

Reflexivity. For every topological space X, the identity map

$$\mathrm{id}_X \colon X \to X$$

is a homotopy equivalence, since

$$\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$$

is homotopic to id_X. Thus X is homotopy equivalent to itself.

Symmetry. Suppose X is homotopy equivalent to Y. Then there exist maps

$$f\colon X \to Y, \qquad g\colon Y \to X$$

such that

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ g \simeq \mathrm{id}_Y.$$

But this is symmetric in f and g, so Y is homotopy equivalent to X.

Transitivity. Suppose X is homotopy equivalent to Y, and Y is homotopy equivalent to Z. Then there exist maps

$$f\colon X \to Y, \quad g\colon Y \to X, \qquad u\colon Y \to Z, \quad v\colon Z \to Y$$

such that

$$g \circ f \simeq \mathrm{id}_X, \quad f \circ g \simeq \mathrm{id}_Y, \qquad v \circ u \simeq \mathrm{id}_Y, \quad u \circ v \simeq \mathrm{id}_Z.$$

We claim that $u \circ f\colon X \to Z$ and $g \circ v\colon Z \to X$ are homotopy inverses.

Indeed,

$$(g \circ v) \circ (u \circ f) = g \circ (v \circ u) \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X,$$

and similarly

$$(u \circ f) \circ (g \circ v) = u \circ (f \circ g) \circ v \simeq u \circ \mathrm{id}_Y \circ v = u \circ v \simeq \mathrm{id}_Z.$$

Hence X is homotopy equivalent to Z.

Therefore homotopy equivalence is an equivalence relation.

Exercise 7.7. A topological space X is said to be contractible if the identity map $\mathrm{id}_X\colon X \to X$ is nullhomotopic. Show that X is contractible if and only if X has the homotopy type of a point.

Solution. Suppose first that X is contractible. Then id_X is nullhomotopic, so there exists a point $x_0 \in X$ such that id_X is homotopic to the constant map

$$c_{x_0}\colon X \to X, \qquad c_{x_0}(x) = x_0.$$

Let $\{*\}$ be a one-point space. Define

$$f\colon X \to \{*\}, \qquad f(x) = *,$$

and

$$g\colon \{*\} \to X, \qquad g(*) = x_0.$$

Then

$$g \circ f = c_{x_0} \simeq \mathrm{id}_X.$$

Also

$$f \circ g = \mathrm{id}_{\{*\}},$$

since every map $\{*\} \to \{*\}$ is the identity. Thus X is homotopy equivalent to a point.

Conversely, suppose X has the homotopy type of a point. Then there exist maps

$$f\colon X \to \{*\}, \qquad g\colon \{*\} \to X$$

such that

$$g \circ f \simeq \mathrm{id}_X.$$

But $g \circ f$ is a constant map $X \to X$, since f maps everything to the unique point of $\{*\}$. Hence id_X is homotopic to a constant map, so X is contractible.

Exercise 7.8. Let X be a topological space and let S^1 be the unit circle considered as a subspace of $\mathbb{R}^2$, where $\mathbb{R}^2$ is given the standard topology. Show that if $f\colon X \to S^1$ is a continuous map that is not surjective, then f is nullhomotopic.

Solution. Since f is not surjective, there exists a point $p \in S^1 \setminus f(X)$. Then

$$f(X) \subseteq S^1 \setminus \{p\},$$

so f factors as

$$X \xrightarrow{\tilde{f}} S^1 \setminus \{p\} \xrightarrow{i} S^1,$$

where i is the inclusion and $\tilde{f}(x) = f(x)$.

Now $S^1 \setminus \{p\}$ is homeomorphic to $\mathbb{R}$, hence contractible. Therefore $\tilde{f}\colon X \to S^1 \setminus \{p\}$ is nullhomotopic: it is homotopic to a constant map

$$c\colon X \to S^1 \setminus \{p\}.$$

Composing this homotopy with the inclusion i, we see that

$$f = i \circ \tilde{f}$$

is homotopic to the constant map $i \circ c\colon X \to S^1$. Hence f is nullhomotopic.