# **Introduction to Topology.**

## Exercise sheet 1: Topological spaces

**Exercise 1.** *Let $X = \{a, b, c\}$. Construct four different topologies on $X$.*

**Exercise 2.** *Let $X$ be a set and $A \subseteq X$. Define the coarsest topology on $X$ such that $A$ is open.*

**Exercise 3.** *Let $X$ be a topological space and suppose that we are given $A, B \subseteq X$. Prove that:*

*i) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

*ii) We have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

**Exercise 4.** *Let $X$ be a topological space and let $A \subseteq X$. Show that $x \in \overline{A}$ if and only if every neighbourhood of $x$ intersects $A$.*

**Exercise 5.** *Show that the rational numbers $\mathbb{Q} \subset \mathbb{R}$ are dense, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$.*

**Exercise 6.** *Let $X$ be a set and consider the cofinite topology on $X$. Given $A \subseteq X$ describe $\mathring{A}$ (the interior of $A$), $\mathrm{Ex}(A)$ (the exterior of $A$) and $\partial A$ (the boundary of $A$).*

**Exercise 7.** *Let $X$ be a metric space and $A \subseteq X$. Prove the following:*

*i) We have $x \in \overline{A}$ if and only if there exists a sequence $\{a_i\}_{i \in \mathbb{N}}$ with each $a_i \in A$ which converges to $x$.*

*ii) Conclude that a map between metric spaces $f : X \to Y$ is continuous if and only if the image of any convergent sequence is again convergent.*

**Exercise 8** ($\star$)**.** *Given an infinite set $X$ show that if we equipp $X$ with the cofinite topology it cannot be homeomorphic to a metric space. What happens if $X$ is finite?*

**Exercise 9** ($\star\star$)**.** *Let* $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ *as defined*[1] *in the lecture. Prove the following:*

*i) Given a metric space* $X$ *then there exists a bijection between the sets*

$$\{\textit{Convergent sequences in } X\} \xleftrightarrow{\ \cong\ } \{\textit{Continuous maps } f : \hat{\mathbb{N}} \to X\}.$$

*ii) Let* $\mathrm{L}(\mathbb{N})$ *be the set of maps*

$$\varphi : \mathbb{N} \longrightarrow \{0, 1\}$$

*such that* $\varphi(m) \leq \varphi(n)$ *if* $m \leq n$*. Show that there is a bijection* $\hat{\mathbb{N}} \cong \mathrm{L}(\mathbb{N})$*.*

*iii) Let* $\mathbb{N}^{\leq a} = \{n \in \mathbb{N} |\ n \leq a\}$ *and consider* $\mathrm{L}(\mathbb{N}^{\leq a})$ *defined in a totally analogous way as* $\mathrm{L}(\mathbb{N})$*. Show that* $\mathrm{L}(\mathbb{N}^{\leq a})$ *consists precisely in* $a + 1$ *elements and that we have continuous maps*

$$r_a : \hat{\mathbb{N}} = \mathrm{L}(\mathbb{N}) \longrightarrow \mathrm{L}(\mathbb{N}^{\leq a})$$

*where* $\mathrm{L}(\mathbb{N}^{\leq a})$ *is equipped with the discrete topology.*

*iv) Show that for* $a < b$ *we maps* $\alpha_{b,a} : \mathrm{L}(\mathbb{N}^{\leq b}) \to \mathrm{L}(\mathbb{N}^{\leq a})$ *such that* $r_a = \alpha_{b,a} \circ r_b$*.*

*vi) Show that given a topological space* $X$ *and a family of continuous maps* $f_a : X \to \mathrm{L}(\mathbb{N}^{\leq a})$ *such that* $f_a = \alpha_{b,a} \circ f_b$ *for every* $a < b$ *then there exists a unique continuous map* $f : X \to \hat{\mathbb{N}}$ *such that for every* $a \in \mathbb{N}$ *we have* $f_a = r_a \circ f$*.*

[1] For ease of notation we will use the convention that $0 \notin \mathbb{N}$