TMA4190: Introduction to Topology.

Final Exam

- Let $n \geq 1$. We will always denote by $\mathbb{R}^n$ the set consisting of n-tuples of real numbers equipped with the standard topology (the one you know from Calculus).
- We let $\mathbb{Q}^n \subset \mathbb{R}^n$ be the subset of those tuples that consist in rational numbers equipped with the subspace topology.
- We denote by $[0,1] = \{x \in \mathbb{R} |\ 0 \leq x \leq 1\} \subset \mathbb{R}$ the closed unit interval, equipped with the subspace topology.
- We denote by $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\} \subset \mathbb{R}^2$ the unit circle, equipped with the subspace topology.

Problem 1. *Define the notion of a topological space and provide the following examples:*

i) Three different topologies on the set $X = \{a, b, c, d\}$.

ii) A topological space which is a metric space.

*iii) A topological space whose topology **does not** come from a metric space. (Remember to provide a proof of your claim).*

Solution. A topological space is a pair (X, τ) consisting in a set X together with a collection of subsets τ of X which we call the open sets, satisfying the following axioms:

1. The subsets $X, \emptyset \in \tau$.

2. Given a set I and a collection $\{X_i\}_{i\in I}$ of subsets such that $X_i \in \tau$ for every $i \in I$ then it follows that the union

$$\bigcup_{i\in I} X_i \in \tau$$

is also an open subset of X.

3. Given a *finite set* N and a collection $\{X_j\}_{j\in N}$ of subsets such that $X_j \in \tau$ then it follows that the (finite) intersection

$$\bigcap_{j\in N} X_j \in \tau$$

is again an open subset of X.

i) We define three topologies on $X = \{a, b, c, d\}$ given by $\tau_1 = \{X, \emptyset\}$, $\tau_2 = \{A \subseteq X \mid A \text{ is a subset of } X\}$ and $\tau_3 = \{X, \emptyset, \{a\}\}$. It is clear that in each of the τ_i for $i = 1, 2, 3$ satisfy the axioms above and thus define three different topologies on X.

ii) Let $X = \mathbb{R}$ and define a metric by $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via the formula $d(x, y) = |x - y|$ (where $|-|$ denotes the absolute value). Let us check that d defines a metric

M1) It follows from the definition of the absolute value that for every $x, y \in \mathbb{R}$ we have $|x - y| \geq 0$ and $|x - y| = 0$ if and only if $x = y$.

M2) It is also immediate from the definition that $|x - y| = |y - x|$ for every $x, y \in \mathbb{R}$.

M3) Finally, given $x, y, z \in \mathbb{R}$ we know from calculus that the triangle inequality holds, in order words, we have

$$|x - z| \leq |x - y| + |y - z|.$$

iii) Let $(\mathbb{R}, \tau_{\text{cof}})$ be a topological space where we declare a subset U (different from $\mathbb{R}$, $\emptyset$) to be open if $\mathbb{R} \setminus U = \{x \in \mathbb{R} | x \notin U\}$ is a finite set. We saw in class that this defines a topology on $\mathbb{R}$ so we need to show that this topology does not come from a metric in $\mathbb{R}$.

Let (X, d) be a metric space and let $x, y \in X$ such that $x \neq y$. Let $d(x, y) = \epsilon$ and note that $\epsilon > 0$. Then it follows that the open balls

$$B\left(x, \frac{\epsilon}{2}\right) \cap B\left(y, \frac{\epsilon}{2}\right) = \emptyset$$

have empty intersection. Indeed, given $z \in B(y, \frac{\epsilon}{2})$ then it follows from the triangle inequality that

$$d(x, y) \leq d(x, z) + d(y, z) \iff d(x, z) \geq d(x, y) - d(y, z) = \epsilon - d(y, z) > \frac{\epsilon}{2}$$

To finish the proof we will show that given $x, y \in \mathbb{R}$ we cannot find open sets $U, V \in \tau_{\text{cof}}$ such that $x \in U$, $y \in V$ and such that $U \cap V = \emptyset$. To see this we observe that since by definition $U = \mathbb{R} \setminus A$ and $V = \mathbb{R} \setminus B$ where A, B are finite sets it follows that

$$U \cap V = \mathbb{R} \setminus A \cup B.$$

Since an union of two finite sets cannot not be equal to $\mathbb{R}$ it follows that $U \cap V \neq \emptyset$.

$\blacksquare$

Problem 2. *Let $A \subset X$ where X is a topological space and such that $A \neq \emptyset$. Let A° be the interior of A, and let $\overline{A}$ be the closure of A. Prove that if $A^{\circ} = \overline{A}$ then A is both open and closed in X.*

Proof. By definition we have that A° is the union of every open set U in X which is contained in A, i.e. $U \subseteq A$. This implies that $A^{\circ} \subseteq A$ and that A° is always open in X.

Dually, $\overline{A}$ is the intersection of all closed subsets $Z \subseteq X$ such that $A \subseteq Z$. This definition implies that $A \subseteq \overline{A}$ and that $\overline{A}$ is always closed in X. Summarizing this we have

$$A^{\circ} \subseteq A \subseteq \overline{A}.$$

If $A^{\circ} = \overline{A}$ then it follows that $A = A^{\circ} = \overline{A}$ and consequently, A is both open and closed in X. $\square$

Problem 3. *Let $A \subset X$ where X is a topological space.*

i) What does it mean for A to be dense in X?

ii) Prove that $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense for $n \geq 1$.

Solution. i) A is dense in X for every open subset $U \subseteq X$ we have that $A \cap U \neq \emptyset$.

ii) Let us suppose that we have proven the statement for $n = 1$ and let us prove the general claim for n. Let $U \subseteq \mathbb{R}^n$ be an open subset. We know that since $\mathbb{R}^n$ is equipped with the product topology there exists some $V \subset U$ such that V is of the form

$$V = \prod_{i=1}^{n} (a_i, b_i) \subset U.$$

Now, by the case $n = 1$ we can pick for each interval (a_i, b_i) a rational number $q_i \in (a_i, b_i)$. It then follows that the tuple $\vec{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{Q}^n$ and that $\vec{q} \in V$. This implies that $U \cap \mathbb{Q}^n \neq \emptyset$.

To show the case $n = 1$ (you might aswell cite that we did it in class) we consider an open interval (a, b) and we wish to show that there exists some $q \in \mathbb{Q}$ such that $q \in (a, b)$. Let $x \in (a, b)$ if $x \in \mathbb{Q}$ we are done. Let us suppose that $x \notin \mathbb{Q}$. Let a_n be the real number whose decimal expression consists in the first n-digits of the decimal expression of x. In particular $a_n \in \mathbb{Q}$. It follows that $|x - a_n| \to 0$ when $n \to \infty$. In particular there must exist some natural number k such that $a_k \in (a, b)$ and thus the claim follows.

$\blacksquare$

Problem 4. *Let $[0,1] \times [0,1]$ and define $C = [0,1] \times [0,1] \big/ \sim$ where the equivalence relation identifies $(0, y) \sim (1, y)$ and $(x, 1) \sim (x', 1)$. Show that C is homeomorphic to $D^2 = \{(a, b) \in \mathbb{R}^2 | a^2 + b^2 \leq 1\}$.*

Solution. We define a map $\Psi : [0,1] \times [0,1] \to D^2$ given by

$$\Psi(x, y) = ((1 - y)\cos(2\pi x), (1 - y)\sin(2\pi x)).$$

this map is clearly continuous (we know it from Calculus). We see that,

$$(1 - y)\cos(2\pi x))^2 + ((1 - y)\sin(2\pi x))^2 = (1 - y)^2 \leq 1$$

so it follows that $\Psi(x, y) \in D^2$.

We wish to show that Ψ descends to C which amounts to checking that Ψ preserves the equivalence relation in the definition of C. To see this we note that

$$\Psi(0,y) = ((1-y)\cos(0),(1-y)\sin(0)) = ((1-y)\cos(2\pi),(1-y)\sin(2\pi)) = \Psi(1,y).$$

Similarly we note that

$$\Psi(x,1) = (0,0) = \Psi(x',1).$$

Consequently we obtain a continuous map, $\overline{\Psi} : C \to D^2$ by the properties of the quotient topology on C.

To finish the proof we need to show that $\overline{\Psi}$ is an homeomorphism. For a fixed value y_0, let i be the inclusion of $i_{y_0} : [0,1]/0 \sim 1 \simeq \mathbb{S}^1 \to C$ sending t to (t, y_0). Then the composition

$$\overline{\Psi} \circ i_{y_0} : \mathbb{S}^1 \to D^2$$

is an homeomorphism into the circle of radius $(1-y_0)^2$. In particular, this shows that $\overline{\Psi}$ is surjective. We observe that we identified every point in $[0,1]\times[0,1]$ where Ψ was not injective. This shows that $\overline{\Psi}$ is injective.

At this point we have a continuous bijective map $\overline{\Psi} : C \to D^2$, so the only thing left to do is to show that its inverse is continuous as well. From the theory, we know that it is enough to show that $\overline{\Psi}$ is a closed map (i.e. it maps closed sets in C to closed sets in D^2).

Let $Z \in C$ be a closed set. Observe that since C is a quotient of a compact space it follows that it is itself compact. Moreover, a closed set of a compact space is (by the theory) again compact. It follows that $\overline{\Psi}(Z)$ is compact in D^2. We know from the theory that a compact subset of a Hausdorff space is itself closed. Since D^2 is Hausdorff it follows that $\overline{\Psi}(Z)$ is again closed and thus $\overline{\Psi}$ is an homeomorphism. ■

Problem 5. *Let $A \subset X$ where X is a topological space. Define the subspace topology on the subset A and prove the following:*

- *A subset $Z \subseteq A$ is closed in the subspace topology if and only if there exists some $Z_X \subseteq X$ which is closed in X such that $Z_X \cap A = Z$.*

Solution. A subset $U \subseteq A$ is open in the subspace topology if and only if there exists some open set $W \subseteq X$ such that $U = W \cap A$.

- Let $Z \subseteq A$ be a closed subset in the subspace topology. Then it follows that $A \backslash Z = U$ is open. Let W be an open set in X such that $W \cap A = U$. We claim that $X \setminus W = K$ is a closed subset such that $K \cap A = Z$. Indeed, we have

$$K \cap A = X \setminus W \cap A = A \setminus (A \cap W) = A \setminus U = Z.$$

The converse follows from the same argument going in the other direction.

■

Problem 6. *Let $K \subset X$ (with $K \neq X$) be a compact subset of a Hausdorff space and let $x \in X \setminus K$. Show that there exists open sets U, V such that $K \subset U$, $x \in V$ and $U \cap V = \emptyset$.*

Solution. For every $y \in K$ let U_y and V_y be open subsets such that $y \in U_y$, $x \in V_y$ and such that $U_y \cap V_y = \emptyset$. The existence of such opens is guaranteed by the fact that X is a Hausdorff space. We know observe that the collection $\{U_y\}_{y \in K}$ is a covering for K. Since K is compact we have a finite collection of points $\{y_0, y_1, \ldots, y_n\}$ such that

$$K \subseteq \bigcup_{i=0}^{n} U_{y_i} = U.$$

We define $V = \bigcap_{i=0}^{n} V_{y_i}$ and observe that $x \in V$ by construction. Moreover, V is an open subset since it is given by a *finite* intersection of open sets. By construction we have

$$U \cap V = \emptyset$$

which finishes the proof. ■

Problem 7. *Let $\mathbf{2} = \{0, 1\}$ be the topological space where we declare the subset $\{0\}$ to be the unique non-trivial open set. Show that the map*

$$H : \mathbf{2} \times [0, 1] \to \mathbf{2},$$

sending an element (x, m) to $H(x, m)$ where the later is given by

$$H(x, m) = \begin{cases} x, & \text{if } m < 1/2 \\ 1, & \text{else} \end{cases}$$

is continuous. Conclude the following:

- *Given a homotopy equivalence $f : X \to Y$ where X is Hausdorff, then in general it is **not true** that Y is Hausdorff.*

Proof. To show that H is continuous we only need to show that $H^{-1}(\{0\})$ is open in $\mathbf{2} \times [0, 1]$. Using the definition of H we find

$$H^{-1}(\{0\}) = \{0\} \times [0, 1/2)$$

since $\{0\}$ is open in $\mathbf{2}$ and $[0, 1/2)$ is open in $[0, 1]$ it follows that $\{0\} \times [0, 1/2) \subset \mathbf{2} \times [0, 1]$ is open in the product topology and consequently we see that H is continuous.

- Let $*$ denote the topological space consisting in one point. It is clear that $*$ is Hausdorff. We claim that $\mathbf{2}$ is homotopy equivalent to $*$. To see this we consider the map $i : * \to \mathbf{2}$ mapping the unique point to 1. Since $i^{-1}(0) = \emptyset$ it follows that i is continuous. Let $t : \mathbf{2} \to *$ be the unique map which is clearly continuous. It is clear that $t \circ i = \mathrm{id}$ is the identity map. The only thing that is left to show is that $i \circ t$ is homotopic to the identity on $\mathbf{2}$. The map H above gives the desired homotopy.

  To finish the proof, we need to show that $\mathbf{2}$ is not Hausdorff. This is clear since we cannot separate the points 0 and 1.

□

Problem 8. *Let $f : X \to Y$ be a homotopy equivalence and suppose that Y is connected. Show that X is connected as well.*

Solution. We assume for contradiction that X is not connected. Let $U, V \subset X$ be two open sets such that $U, V \neq X, \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$.

Given a connected topological space Z and a continuous map $h : Z \to X$ it follows that $h(Z)$ is either in U or in V since otherwise $h^{-1}(U)$ and $h^{-1}(V)$ would give a separation for Z which contradicts the fact that Z is connected. In particular, given points $x \in U$ and $y \in V$ it follows that since $[0, 1]$ is connected there cannot exist a path connecting x and y.

The fact f is a homotopy equivalence implies that there exists a map $g : Y \to X$ and a homotopy between $g \circ f$ and the identity map on X. Let us assume without loss of generality that $g(Y) \subset U$. Given $x \in V$ it follows from the existence of the homotopy $g \circ f \sim \mathrm{id}_X$ that there is a path between $g \circ f(x) \in U$ and $x \in V$ which is a contradiction by the discussion above. ■

Problem 9. *What does it mean for a continuous map $f : X \to Y$ to be open? Give an example of an open map.*

Proof. A map is open if given an open set $U \subseteq X$ then it follows that $f(U)$ is open in Y. Let X, Y be topological spaces and let $X \times Y$ be the cartesian product. We consider the map $\pi_X : X \times Y \to X$ given by $\pi_X = (x, y) = x$ and claim that π_X is open.

We know from theory that any open $U \subset X \times Y$ can be expressed as

$$U = \bigcup_{i \in I} A_i \times B_i,$$

where $A_i \subseteq X$ is open and similarly $B_i \subseteq Y$ is also open. Then it follows that $\pi_X(U) = \bigcup_{i \in I} A_i$ which is an union of open sets of X and thus is open. $\square$

Problem 10. *Let* $\mathrm{Mo} = [0,1] \times [0,1]/\sim$ *where the equivalence relation identifies* $(0, x) \sim (1, 1 - x)$. *Show that* Mo *is path connected and compute* $\pi_1(\mathrm{Mo}, x)$.

Hint: $\mathbb{S}^1 \simeq \{(x, y) \in \mathrm{Mo}\,|y = 1/2\} \subset \mathrm{Mo}$

Solution. First let us show that Mo is path connected. We consider the (continuous) quotient map

$$\pi : [0,1] \times [0,1] \to \mathrm{Mo}$$

and observe that π is surjective. Given $x, y \in \mathrm{Mo}$ we pick a pair of points $a, b \in [0,1] \times [0,1]$ such that $\pi(a) = x$ and $\pi(b) = y$. We note that we can construct a path $\gamma : [0,1] \to [0,1] \times [0,1]$ joining a and b. This is true since we can connect points in $[0,1] \times [0,1]$ using straight lines. To finish the proof we observe that $\pi \circ \gamma$ is the desired path.

The final thing to do is to compute $\pi_1(\mathrm{Mo}, x)$. We will do this by showing that the map in the hint defines a homotopy equivalence. Then it will follow that $\pi_1(\mathrm{Mo}, x) \simeq \pi_1(\mathbb{S}^1, x) \simeq \mathbb{Z}$. We define a map

$$T : [0,1] \times [0,1] \to \mathbb{S}^1 = [0,1]/0 \sim 1$$

which sends (s, x) to s. The map T is clearly continuous. Now we show that T preserves the equivalence relation. This follows easily after noting that

$$T(0, x) = 0 \sim 1 = T(1, 1 - x).$$

We conclude that we have a continuous map $\overline{T} = \mathrm{Mo} \to \mathbb{S}^1$. Let $i : \mathbb{S}^1 \to \mathrm{Mo}$ be the map in the hint. Then it is clear that $\overline{T} \circ i$ is the identity map on $\mathbb{S}^1$. To finish the proof we construct a homotopy

$$H : \mathrm{Mo} \times [0,1] \to \mathrm{Mo}$$

between the identity map on Mo and $i \circ T$. We define for $z \in \mathrm{Mo}$ with $z = (x, y)$

$$H(z,t) = (x, t/2 + (1-t)y).$$

Observe that we have $H(z,0) = (x,y)$ and $H(z,1) = (x, 1/2)$ so the final thing to check is that H is a well defined continuous map. This amounts to showing that H respect the equivalence relation in the definition of Mo. Let $z = (0, y)$ then we have

$$H(z,t) = (0, t/2 + (1-t)y) \sim (1, 1 - (t/2 + (1-t)y)),$$

on the other hand, given $z' = (1, 1-y)$ we have

$$H(z',t) = (1, t/2 + (1-t)(1-y)),$$

we only need to check that

$$1 - \frac{t}{2} + (t-1)y = \frac{t}{2} + (1-t)(1-y) \iff$$

$$1 - \frac{t}{2} + (t-1)y = \frac{t}{2} + (1-t) + (t-1)y \iff 1 - \frac{t}{2} = \frac{t}{2} + (1-t).$$

■