INTRODUCTION TO TOPOLOGY

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Contents

1. Introduction

These are lecture notes from the course TMA4190 Introduction to Topology given in the Spring semester 2021 at NTNU. They are intended as a supplement to the lectures and may not be entirely self-contained.

Please send me an email if you spot any errors!

What is topology?

Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone. . .

— Aleksandr Solzhenitsyn

Topology is a part of mathematics concerned with the study of spaces. In topology, we consider two spaces to be *equivalent* if one can be continuously deformed into the other. Such a continuous deformation is known as a *homeomorphism*, i.e., a continuous bijection with a continuous inverse. See Figure 1.1 for an example of two homeomorphic spaces.

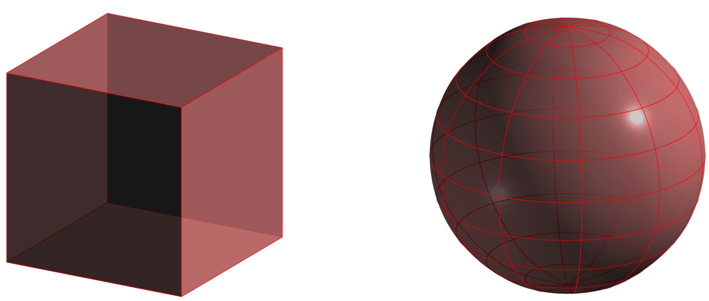


Figure 1.1: The surface of the (unit) cube and the (unit) sphere S^2 are homeomorphic.

We might ask ourselves the following question.

Question Let X and Y be two spaces. Does there exist a homeomorphism $\varphi\colon X \to Y$? In other words, are X and Y homeomorphic?

Showing that two spaces are homeomorphic involves the construction of a specific homeomorphism between them. Proving that two spaces are *not* homeomorphic is a problem of a different nature. It is a hopeless exercise to check every possible map between the two spaces for whether or not it is a homeomorphism. Instead we might check to see whether there is some "topological invariant" of spaces (where this invariant is preserved under a homeomorphism) that allows us to differentiate between the two spaces.

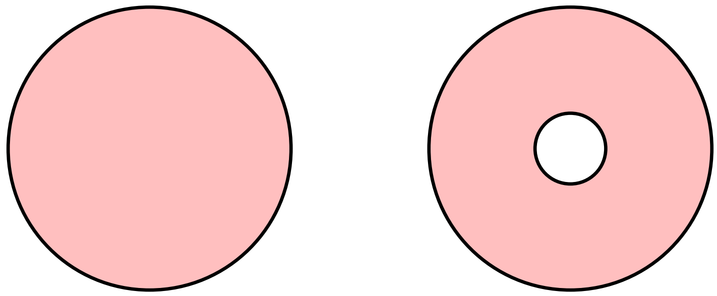


Figure 1.2: The disc D^2 and the annulus are not homeomorphic.

One instrument to help us detect topological information of a space is the *fundamental group* associated to the space. It is reasonable to expect that the disc D^2 and the annulus are not homeomorphic. The annulus has a hole through it while the disc does not, see Figure 1.2.

To detect the hole through the annulus we may use loops, i.e., continuous maps from the unit interval to the annulus with the endpoints identified. See Figure 1.3.

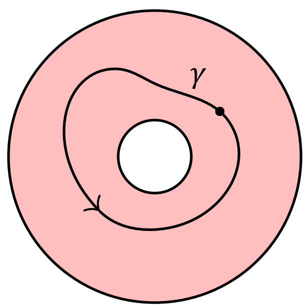


Figure 1.3: A loop.

It is then possible to construct a *group* involving such loops. This group is what is known as the fundamental group.

Some applications

To help illustrate some of the power of topology, let us consider two theorems, both of which may be proved using topology and, more specifically, the fundamental group.

The first theorem is the Brouwer fixed point theorem.

Theorem 1.1 **(Brouwer fixed point theorem)** *Let $f: D^n \to D^n$ be a continuous map from the (unit) disk in $\mathbb{R}^n$ to itself. Then f has a fixed point, i.e., there is some point $x \in D^n$ such that $f(x) = x$.*

For $n = 1$ this is a well-known result from calculus: The graph of any continuous map $f: [0,1] \to [0,1]$ must cross the diagonal $y = x$ for some $x_* \in [0,1]$. Hence, $f(x_*) = x_*$. See Figure 1.4.

The second theorem is the fundamental theorem of algebra.

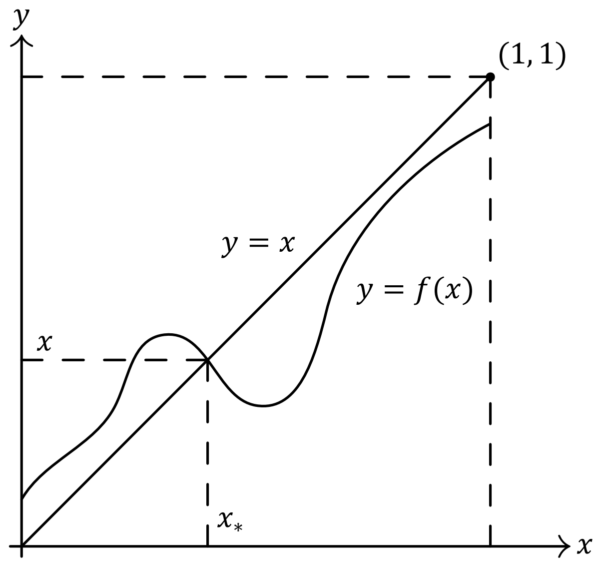


Figure 1.4: The graph of any continuous map from $[0, 1]$ to $[0, 1]$ must cross the diagonal.

Theorem 1.2 **(The fundamental theorem of algebra)** *A polynomial equation*

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

of degree $n > 0$ with complex coefficients has at least one complex root.

To prove it we will use the fact that the fundamental group of the circle is isomorphic to the group of integers. The fundamental theorem of algebra may be proved in many different ways, including using only algebraic techniques and analysis. However, the proof we will provide (based on [4, §56]) is a fairly simple corollary of the computation of the fundamental group of the circle.

2. Continuous maps

2.1 Metric spaces

From calculus we know what to mean by a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$: a map $f: \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $p \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|p - q\|_{\mathbb{R}^n} < \delta$, then $\|f(p) - f(q)\|_{\mathbb{R}^m} < \epsilon$. Here $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. Similarly, $\|\cdot\|_{\mathbb{R}^m}$ denotes the Euclidean norm in $\mathbb{R}^m$.

Topological spaces provide the most general setting for which the concept of continuity makes sense. Before we get to the concept of a topological space, let us consider metric spaces. Metric spaces allow us to speak of distance between elements. Using the notion of distance between elements we can make sense of continuity of maps between metric spaces.

Definition 2.1 **(Metric spaces)** A *metric space* (X, d) is a non-empty set X together with a map $d: X \times X \to \mathbb{R}$ called a *metric* such that the following properties hold:

M1 $d(x, y) \geqslant 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

M2 $d(x, y) = d(y, x)$ for all $x, y \in X$;

M3 $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The first condition says that the distance between two elements is always positive, and equal to zero if and only if the two elements are the same. The second condition says that distance is symmetric. The third condition says that the *triangle inequality* holds. The metric d is sometimes also referred to as a distance function.

Example 2.2 **($\mathbb{R}^n$ seen as a metric space)** Let $X = \mathbb{R}$ and d be the map defined by $d(x, y) = |x - y| (= \sqrt{(x-y)^2})$. The first two requirements for d are clearly satisfied, and the third follows from the usual triangle inequality for real numbers,

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leqslant |x - y| + |y - z| = d(x, y) + d(y, z).$$

For $X = \mathbb{R}^n$ with $n > 0$ an integer, let $d(x, y) = \|x - y\|$ where $\|\cdot\|$ is the Euclidean norm, e.g., for $n = 2$, $d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Again, the first two requirements for d are clearly satisfied. The third requirement follows from the triangle inequality for vectors in $\mathbb{R}^n$.

We may equip $\mathbb{R}^n$ with other metrics than the one described in Example 2.2. For instance, for $X = \mathbb{R}^2$, let

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is known as the *taxi cab metric*.

We say that two metrics d_1 and d_2 on the same set X are *equivalent* if there are constants L and M such that

$$d_1(x,y) \leqslant Ld_2(x,y) \quad \text{and} \quad d_2(x,y) \leqslant Md_1(x,y)$$

for all $x, y \in X$.

Example 2.3 (Discrete metric spaces) For any set X, let $d\colon X \times X \to \mathbb{R}$ be the map given by

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y. \end{cases}$$

We call d the *discrete metric* on X.

Example 2.4 ($C[a,b]$) Let $X = C[a,b]$, i.e., the set of continuous maps from the interval $I = [a,b] \subseteq \mathbb{R}$ to $\mathbb{R}$, and let

$$d(x,y) = \max_{i \in I} |x(i) - y(i)|.$$

Example 2.5 If d is a metric on a set X, and $A \subseteq X$ is any subset of X, then d is also a metric on A.

2.2 Continuous maps between metric spaces

The definition of continuity of maps between metric spaces is completely analogous to the situation that we have from calculus.

Definition 2.6 (Continuous maps between metric spaces) Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is *continuous* at $p \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(p,q) < \delta$ then $d_Y(f(p), f(q)) < \epsilon$.

If f is continuous at every $p \in X$, we say that f is *continuous*.

To get us to the setting of topological spaces we will need the concept of *open* and *closed* sets.

Definition 2.7 (Open and closed balls) Let (X,d) be a metric space, and let $a \in X$ and $r > 0$ be real number. The *open ball* centered at a with radius r is the subset

$$\mathrm{B}(a;r) = \{x \in X \mid d(x,a) < r\}$$

of X. The *closed ball* centered at a with radius r is the subset

$$\overline{\mathrm{B}}(a;r) = \{x \in X \mid d(x,a) \leqslant r\}$$

of X.

In Euclidean space with the usual metric (induced from Euclidean norm), a ball (as defined above) is precisely what we think of as a ball in everyday language. Open balls are sometimes referred to as simply balls, and closed balls are sometimes referred to as discs, e.g. Theorem 1.1.

Example 2.8 (Open balls in discrete metric spaces) Let (X, d) be the metric space defined in Example 2.3. Then

$$\mathrm{B}(x; r_1) = \{x\} \quad \text{and} \quad \mathrm{B}(x; r_2) = X$$

for all $0 < r_1 \leqslant 1$ and all $r_2 > 1$.

Definition 2.9 (Open and closed sets) Let (X, d) be a metric space. A subset $A \subseteq X$ is *open* in X if for every point $a \in A$, there is an open ball $\mathrm{B}(a; r)$ about a contained in A. We say that A is *closed* in X if the complement $A^c = X \setminus A = \{x \in X \mid x \notin A\}$ is open.

Most subsets are *neither* open nor closed. Subsets that are both open and closed are sometimes referred to as *clopen*. In particular, both $\emptyset$ and X are clopen in X.

Lemma 2.10 *Let (X, d) be a metric space, $x \in X$ and $r > 0$ a real number. Then the open ball $\mathrm{B}(x; r) \subseteq X$ is open in X, and the closed ball $\overline{\mathrm{B}}(x; r) \subseteq X$ is closed in X.*

Proof. We prove the statement about open balls. The statement about closed balls follows from a similar argument.

Assume that $y \in \mathrm{B}(x; r)$. We need to prove that there is an open ball $\mathrm{B}(y; \epsilon)$ about y that is contained in $\mathrm{B}(x; r)$. Let $\epsilon = r - d(x, y)$. By the triangle inequality of the metric d, M3, we have that for $z \in \mathrm{B}(y; \epsilon)$,

$$d(x, z) \leqslant d(x, y) + d(y, z) < d(x, y) + \epsilon = d(x, y) + r - d(x, y) = r.$$

Hence, $\mathrm{B}(y; \epsilon) \subseteq \mathrm{B}(x; r)$. □

For a metric space (X, d), a subset $A \subseteq X$ and $x \in X$, we say that: (i) x is an *interior point* of A if there is an open ball $\mathrm{B}(x; r)$ about x which is contained in A, (ii) x is an *exterior point* of A if there is an open ball $\mathrm{B}(x; r)$ which is contained in A^c and (iii) x is a *boundary point* if all open balls about x contains points in A and in A^c. Hence, A is open in X if and only if A only consists of its interior points. An interior point will *always* belong to A. An exterior point will *never* belong to A. A boundary point will some times belong to A, and some times to A^c.

Definition 2.11 (Neighborhoods) Let (X, d) be a metric space, A a subset of X and $x \in X$. We say that A is a *neighborhood of x* if there is an open ball about x that is contained in A. We say that A is an *open neighborhood* (of x) if A itself is open.

Theorem 2.12 (Continuity at a point) *Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $p \in X$. A map $f : X \to Y$ is continuous at p if and only if for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$.*

Proof. Assume that f is continuous at p. If B is a neighborhood of $f(p)$, then, by definition, there is an open ball $\mathrm{B}_Y(f(p);\epsilon)$ about $f(p)$ that is contained in B. Since f is continuous at p, there is a $\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. Hence, $f(\mathrm{B}_X(p;\delta)) \subseteq \mathrm{B}_Y(f(p);\epsilon) \subseteq B$. That is, if we let $A = \mathrm{B}_X(p,\delta)$, then for all neighborhoods B of $f(p)$, we have that $f(A) \subseteq B$ where A is a neighborhood of p.

Assume that for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$. We need to prove that for all $\epsilon > 0$, there is a $\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. By utilizing the fact that $B = \mathrm{B}_Y(f(p);\epsilon)$ is a neighborhood of $f(p)$, then, by assumption, there must be a neighborhood A of p such that $f(A) \subseteq B$. Since A is a neighborhood of p, there is an open ball $\mathrm{B}_X(p;\delta)$ about p that is contained in A. Now assume that $d_X(p,p') < \delta$. Then $p' \in \mathrm{B}_X(p;\delta) \subseteq A$. Thus $f(p') \in B = \mathrm{B}_Y(f(p);\epsilon)$, and hence, $d_Y(f(p), f(p')) < \epsilon$. Thus f is continuous at p. □

The following theorem gives an alternative description of continuous maps between metric spaces.

Theorem 2.13 (Continuous maps between metric spaces) *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is continuous if and only if for every subset $B \subseteq Y$ open in Y, the* preimage *of B under f,*

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X,$$

is open in X.

Proof. Assume that f is continuous. For $B \subseteq Y$ open in Y, we have to prove that $f^{-1}(B) \subseteq X$ is open in X. Let $a \in f^{-1}(B)$. We want to prove that there is an open ball about a in X that is contained in $f^{-1}(B)$. By assumption, $B \subseteq Y$ is open in Y. Hence, there is an $\epsilon > 0$ such that $\mathrm{B}_Y(f(a);\epsilon) \subseteq B$. From the assumption that f is continuous there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq f^{-1}(B)$.

We now prove the opposite implication. Assume that for every subset $B \subseteq Y$ open in Y, the preimage $f^{-1}(B)$ of B under f is open in X. Let $a \in X$ and $\epsilon > 0$ be a real number. From the first assumption it follows that $f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq X$ is open in X. As $f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$ is open and contains a, there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$. Thus $x \in \mathrm{B}_X(a;\delta)$ implies that $f(x) \in \mathrm{B}_Y(f(a);\epsilon)$. Hence, $f\colon X \to Y$ is continuous. □

Let A and B be sets, and let $f\colon A \to B$. Then $f^{-1}(B)$ will always exist even if there is no inverse map. In the cases where f has an inverse there is no ambiguity. If U and V are both subsets of B then

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \quad \text{and} \quad f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V),$$

and furthermore, if $U \subseteq V$ then $f^{-1}(U) \subseteq f^{-1}(V)$. Let $U \subseteq A$ and $V \subseteq B$, then

$$U \subseteq f^{-1}(f(U)) \quad \text{and} \quad f(f^{-1}(V)) \subseteq V.$$

We also note that if U is a subset of B then

$$f^{-1}(B \setminus U) = f^{-1}(U^c) = (f^{-1}(U))^c = A \setminus f^{-1}(U).$$

2.3 Exercises

Exercise 2.1 Does $d(x, y) = (x - y)^2$ define a metric on $X = \mathbb{R}$?

Exercise 2.2 Show that $\mathbb{R}^2$ equipped with the taxi cab metric is a metric space.

Exercise 2.3 Let (X, d) be a metric space. Show that the map $d' : X \times X \to \mathbb{R}$ given by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

Exercise 2.4 Draw a picture of the open ball $\mathrm{B}((0, 0); 1)$ in the metric space $(\mathbb{R}^2, d)$ with

(a) $d(x, y) = d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$;

(b) $d(x, y) = d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$;

(c) $d(x, y) = d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Exercise 2.5 Show that d_1, d_2 and d_∞ (as defined in Exercise 2.4) are equivalent on $X = \mathbb{R}^2$.

Exercise 2.6 Show that in a discrete metric space (X, d), cf. Example 2.3, every subset is both open and closed in X.

Exercise 2.7 Show that for equivalent metrics d and d' on the set X, the open sets are the same.

Exercise 2.8 Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a map. Show that f is continuous if and only if for every subset $B \subseteq Y$ closed in Y, the preimage $f^{-1}(B)$ is closed in X.

3. Topological spaces

3.1 Definition and examples

Topological spaces are spaces constructed to support continuous maps. The definition is as follows.

Definition 3.1 (Topological spaces) A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X that are called *open* in X, such that the following properties hold.

T1 The subsets $\emptyset$ and X are in $\mathcal{T}$.

T2 The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

T3 The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is strictly speaking an ordered pair $(X, \mathcal{T})$. We refer to $\mathcal{T}$ as the *topology* on X. We will often omit specific mention of $\mathcal{T}$ if no confusion will arise.

The following theorem states that every metric space (X, d) is a topological space with the *metric topology* $\mathcal{T}_d$ on X.

Theorem 3.2 (Metric spaces are topological spaces) *Let (X, d) be a metric space. Let $\mathcal{T}_d$ be the collection of subsets $U \subseteq X$ with the property that $U \in \mathcal{T}_d$ if and only if for each $x \in U$ there is an $r > 0$ such that $\mathrm{B}(x; r) \subseteq U$. Then $\mathcal{T}_d$ defines a topology on X.*

Proof. Clearly, $\emptyset \in \mathcal{T}_d$. To show that $X \in \mathcal{T}_d$, note that for any $x \in X$, $\mathrm{B}(x; 1) \subseteq X$. Hence, $X \in \mathcal{T}_d$. Thus T1 is satisfied.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be any subcollection of $\mathcal{T}_d$. We need to prove that $V = \bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_d$. Let $x \in V$. From $V = \bigcup_{\lambda \in \Lambda} U_\lambda$ there is $\lambda \in \Lambda$ such that $x \in U_\lambda$. By the property of U_λ satisfied by the U_λ in $\mathcal{T}_d$ there is an $r > 0$ such that $\mathrm{B}(x; r) \subseteq U_\lambda$. Hence, $\mathrm{B}(x; r) \subseteq V$. Thus $V \in \mathcal{T}_d$, and T2 is satisfied.

We prove that the intersection of two elements of $\mathcal{T}_d$ is also an element of $\mathcal{T}_d$. The general result then follows by an induction argument. Let $U_1, U_2 \in \mathcal{T}_d$. We need to prove that $U_1 \cap U_2 \in \mathcal{T}_d$. Let $x \in U_1 \cap U_2$. Since $U_1 \cap U_2 \subseteq U_i$, we have that $x \in U_i$ for $i = 1, 2$. By the defining property of $\mathcal{T}_d$ there is an $r_i > 0$ such that $\mathrm{B}(x; r_i) \subseteq U_i$ for $i = 1, 2$. Let $r = \min\{r_1, r_2\}$. Then $\mathrm{B}(x; r) \subseteq \mathrm{B}(x; r_i) \subseteq U_i$ for $i = 1, 2$. Thus $\mathrm{B}(x; r) \subseteq U_1 \cap U_2$, and so $U_1 \cap U_2 \in \mathcal{T}_d$. Hence, T3 is satisfied. $\square$

The following theorem relates the metric topologies for two equivalent metrics.

Theorem 3.3 *Let X be any set, and let d_1 and d_2 be two equivalent metrics on X. Then $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.*

This follows from Exercise 2.7.

Example 3.4 (Discrete topology) Let X be any set. The collection $\mathcal{T}$ of all subsets of X, i.e. the power set $\mathcal{P}(X)$ of X, is a topology on X. We refer to this topology as the *discrete topology*. A set X equipped with the discrete topology is referred to as a *discrete topological space*.

The discrete topology is the unique topology where the singletons are open. We can think of a discrete topological space as a space of separate, isolated points, with no close interaction between different points.

For any set X, the discrete topology is the largest topology we may equip X with. The smallest topology is called the indiscrete topology.

Example 3.5 (Indiscrete topology) Let X be any set. The collection $\mathcal{T}$ consisting of $\emptyset$ and X is a topology on X, referred to as the *indiscrete topology* on X. A set X equipped with the indiscrete topology is referred to as an *indiscrete topological space*.

Example 3.6 Let $X = \{a, b, c\}$. The following collections all define a topology on X.

(1) $\mathcal{T}_1 = \mathcal{T}_{\text{ind}} = \{\emptyset, X\}$

(2) $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$

(3) $\mathcal{T}_3 = \{\emptyset, \{a, b\}, X\}$

(4) $\mathcal{T}_4 = \{\emptyset, \{a\}, \{a, b\}, X\}$

(5) $\mathcal{T}_5 = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, X\}$

(6) $\mathcal{T}_6 = \mathcal{T}_{\text{disc}} = \mathcal{P}(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}, \{c\}, X\}$

There are in total 29 topologies on X. However, there are also collections of subsets of X which do not define topologies on X. None of the following collections of subsets of X define a topology on X.

(1) $\{\emptyset, \{a\}, \{b\}, X\}$

(2) $\{\emptyset, \{a\}, \{c\}, X\}$

(3) $\{\emptyset, \{a, b\}, \{b, c\}, X\}$

Definition 3.7 (Comparable topologies) Let X be any set and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are *comparable* if either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Clearly, for any set X, the discrete topology $\mathcal{T}_{\text{disc}}$ contains the indiscrete topology $\mathcal{T}_{\text{ind}}$: $\mathcal{T}_{\text{disc}} \supseteq \mathcal{T}_{\text{ind}}$. Hence, the discrete topology is finer than the indiscrete topology and the indiscrete topology is coarser than the discrete topology.

Example 3.8 (Cofinite topology) Let X be any set. The collection $\mathcal{T}$ of subsets of X consisting of subsets $U \subseteq X$ such that $U^c = X \setminus U$ is either finite or all of X is a topology on X. We refer to this topology as the *cofinite topology* on X.

If X is a finite set, the cofinite topology is equal to the discrete topology. If X is an infinite set, the cofinite topology is *strictly coarser* than the discrete topology, in the sense that the cofinite topology is properly contained in the discrete topology.

We end this section with a theorem that we can use to prove that some set is open. To state the theorem we need the following definition.

Definition 3.9 (Neighborhoods) Let X be a topological space, U a subset of X and $x \in X$. We say that U is a *neighborhood of* x if $x \in U$ and U is open in X.

A neighborhood in the sense of the previous definition is sometimes referred to as an *open* neighborhood, cf. Definition 2.11.

Theorem 3.10 *Let X be a topological space. A subset U of X is open in X if and only if for every $x \in U$ there is a neighborhood U_x of x such that $U_x \subseteq U$.*

Proof. Assume that U is open in X. Then for every $x \in U$, U is a neighborhood of x that is contained in U.

We prove the other implication. Assume that for every $x \in U$ there is a $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq U$, i.e., that U_x is a neighborhood of x such that $U_x \subseteq U$. To prove that $U \in \mathcal{T}$, we will prove that $U = \bigcup_{x \in U} U_x$. Assume that $x' \in U_{x'}$. Then $x' \in U_{x'} \subseteq \bigcup_{x \in U} U_x$. Furthermore, any point in $\bigcup_{x \in U} U_x$ is in U_x for some $x \in U$ so by assumption, $U_x \subseteq U$ and $x \in U_x \subseteq U$. Hence, $U = \bigcup_{x \in U} U_x$. As U is the union of open sets it must be an open set as well by T2. □

3.2 Continuous maps

We know from Theorem 2.13 that a map between metric spaces is continuous if and only if the preimage of an open set is open. This motivates the following definition.

Definition 3.11 (Continuous maps between topological spaces) Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous*if preimages of open sets are open, i.e., if V is an open set in Y then the preimage $f^{-1}(V)$ of V under f is open in X.

Hence, all continuous maps between metric spaces (X, d_X) and (Y, d_Y) are also continuous maps between the corresponding topological spaces X and Y with the metric topologies $\mathcal{T}_{d_X}$ and $\mathcal{T}_{d_Y}$, respectively.

Example 3.12 Let X and Y be topological spaces. Then all constant maps from X to Y are continuous: the preimages are either empty or the entire space, and these are always open, cf. T1.

Example 3.13 Let X be a discrete topological space and Y a topological space. Then all maps from X to Y are continuous.

Example 3.14 Let X be any topological space and Y be an indiscrete topological space. Then all maps from X to Y are continuous.

Example 3.15 Let $\mathbb{R}_{\text{disc}}$ be the discrete topological space consisting of the real numbers with the discrete topology, and let $\mathbb{R}$ be the topological space consisting of the the real numbers with the usual (Euclidean) metric topology. Then the identity map

$$\mathbb{R}_{\text{disc}} \xrightarrow{\text{id}} \mathbb{R}$$

is continuous by Example 3.13, while the identity map

$$\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}_{\text{disc}}$$

is *not* continuous: singletons are open in the discrete topology but not in the (Euclidean) metric topology.

The following theorem says that the composition of two continuous maps is a continuous map.

Theorem 3.16 **(Composition of continuous maps)** *Let X, Y and Z be topological spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then the composite $g \circ f\colon X \to Z$ is continuous.*

Proof. Let $W \subseteq Z$ be open in Z. We need to prove that $(g \circ f)^{-1}(W)$ is open in X. Since

$$\begin{aligned}(g \circ f)^{-1}(W) &= \{x \in X \mid g(f(x)) \in W\} \\ &= \{x \in X \mid f(x) \in g^{-1}(W)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(W))\} = f^{-1}(g^{-1}(W))\end{aligned}$$

and that $g^{-1}(W)$ is open in Y and $f^{-1}(g^{-1}(W))$ is open in X (by continuity of g and f), it follows that $(g \circ f)^{-1}(W)$ is open in X. Hence, $g \circ f$ is continuous. □

We can express continuity at a point for maps between topological spaces using neighborhoods. (See Theorem 2.12 for the case of metric spaces.)

Definition 3.17 **(Continuity at a point)** Let X and Y be topological spaces, and let $x \in X$. A map $f\colon X \to Y$ is *continuous at x* if for all neighborhoods V of $f(x)$ there is a neighborhood U of x such that $f(U) \subseteq V$.

Theorem 3.18 *Let X and Y be topological spaces. A map $f\colon X \to Y$ is continuous if and only if it is continuous at each $x \in X$.*

Proof. Assume that f is continuous, and let $x \in X$ and V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subseteq V$.

Assume that f is continuous at each $x \in X$. Let $V \subseteq Y$ be open in Y. Choose $x \in f^{-1}(V)$. Since f is continuous at x there is neighborhood U_x of x such that $f(U_x) \subseteq V$. Hence, $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, and hence, it is open in X. Thus f is continuous. □

3.3 Homeomorphisms

We now introduce the notion of topological equivalence, also known as homeomorphism.

Definition 3.19 **(Homeomorphisms)** Let X and Y be topological spaces. A bijective map $f\colon X \to Y$ with the property that both f and $f^{-1}\colon Y \to X$ are continuous, is called a *homeomorphism*. If there exists a homeomorphism $f\colon X \to Y$, we say that X and Y are *homeomorphic* and write $X \cong Y$.

A homeomorphism $f\colon X \to Y$ gives a one-to-one correspondence between open sets in X and Y. As a result, any property of a topological space that can be expressed in terms of its elements and its open subsets is preserved by homeomorphisms. Such a property is called a *topological property*.

Example 3.20 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 2x - 1$$

is a homeomorphism. Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(y) = \frac{1}{2}(y + 1)$$

then, clearly, $g(f(x)) = x$ and $f(g(y)) = y$ for all real numbers x and y. Thus f is a bijection and $f^{-1} = g$. From calculus we know that f and g are continuous. Hence, f is a homeomorphism.

Example 3.21 Let $X = \{a, b\}$, and let $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$ be two topologies on X. The map $f\colon X \to X$ given by $f(a) = b$ and $f(b) = a$ is clearly a continuous bijection (with the domain given $\mathcal{T}_1$ as topology, and the codomain given $\mathcal{T}_2$ as topology). Also, f is its own inverse: $f = f^{-1}$. Hence, f is a homeomorphism and $(X, \mathcal{T}_1) \cong (X, \mathcal{T}_2)$.

Homeomorphisms are continuous bijections, but the converse is not true.

Example 3.22 Let $X = \{a, b\}$. The identity map $\mathrm{id}\colon X \to X$ where the domain is given the discrete topology and the codomain is given the indiscrete topology is a continuous bijection but *not* a homeomorphism: the inverse map is not continuous.

The following theorem says that being homeomorphic is an equivalence relation on any set of topological spaces.

Theorem 3.23 *Let X, Y and Z be topological spaces.*

Reflexivity *The identity map* $\mathrm{id}\colon X \to X$ *(where the domain and the codomain are equipped with the same topology), given by* $\mathrm{id}(x) = x$ *for* $x \in X$*, is a homeomorphism.*

Symmetry *If* $f\colon X \to Y$ *is a homeomorphism, then* $f^{-1}\colon Y \to X$ *is also a homeomorphism.*

Transitivity *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are homeomorphisms, then* $g \circ f\colon X \to Z$ *is also a homeomorphism.*

Proof. The identity map $\mathrm{id}\colon X \to X$ (where the domain and the codomain are equipped with the same topology) is clearly continuous and bijective. As the identity map is its own inverse, then it is also a homeomorphism. Hence, $X \cong X$ and so $\cong$ satisfies the reflexivity condition for an equivalence relation.

If $f\colon X \to Y$ is a homeomorphism, then $f^{-1}\colon Y \to X$ is also a homeomorphism: f^{-1} is a continuous bijection with continuous inverse $(f^{-1})^{-1} = f\colon X \to Y$. Hence, $X \cong Y$ if and only if $Y \cong X$. Thus $\cong$ satisfies the symmetry condition for an equivalence relation.

Theorem 3.16 tells us that the composition of two homeomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is continuous. The composition of two bijective maps is always bijective. Hence, $g \circ f$ is a continuous bijection. We need to prove that its inverse, $(g \circ f)^{-1}$, is continuous. Since

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

is a composition of continuous maps, then by Theorem 3.16 so is $(g \circ f)^{-1}$. Thus $g \circ f$ is a homeomorphism. Hence, if $X \cong Y$ and $Y \cong Z$, then $X \cong Z$. Thus $\cong$ satisfies the transitivity condition for an equivalence relation. □

3.4 Closed sets

Recall that in a topological space X, a subset A of X is an open subset if and only if A is an element of the topology of X, i.e., $A \in \mathcal{T}$.

Definition 3.24 **(Closed subsets)** A subset K of a topological space X is *closed* in X if and only if the complement $K^c = X \setminus K$ is open in X.

This is completely analogous to how we defined closed subsets in metric spaces, cf. Definition 2.9.

Example 3.25 Let X be a discrete topological space. Since every subset of X is open in X, it follows that every subset of X is also closed in X.

Example 3.26 Let X be an indiscrete topological space. The only subsets of X that are closed in X are $\emptyset$ and X (which are also the only subsets that are open in X).

Recall that in a discrete topological space, all the singletons are open sets. This is usually *not* the case.

Example 3.27 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Then every subset $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\} \subseteq \mathbb{R}$ is closed in $\mathbb{R}$: the complement $[a, b]^c = \mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of open sets in $\mathbb{R}$, and hence, is open in $\mathbb{R}$. Furthermore, all the singletons are closed: the complement $\{a\}^c = \mathbb{R}\setminus\{a\} = (-\infty, a) \cup (a, \infty)$ is a union of open sets in $\mathbb{R}$, and hence, is open in $\mathbb{R}$.

By passing to complements we get the following theorem.

Theorem 3.28 *Let X be a topological space.*

(1) *Both $\emptyset$ and X are closed (as subsets) in X.*

(2) *The intersection of any subcollection of closed sets in X is closed in X.*

(3) *The union of any finite subcollection of closed sets in X is closed in X.*

It follows that we could have defined a topological space X by specifying a collection of subsets of X satisfying the three statements in Theorem 3.28 where we would say that a subset of X is open in X if its complement is closed in X.

We end this section with a theorem describing the connection between continuous maps and closed sets. We will need the following definition.

Definition 3.29 (Closure) Let X be a topological space, and let A be a subset of X. The *closure of A*, written $\overline{A}$, is the intersection of all subsets of X that contain A and which are closed in X.

From the definition it follows that $\overline{A}$ is the smallest subset of X that contains A and which is closed in X. Furthermore, if A is closed in X, then $\overline{A} = A$.

There is an analogous definition for open sets where we take union instead of intersection. We can define the *interior of A*, written $\operatorname{Int}(A)$, to be the union of all subsets of X that are contained in A and which are open in X. It follows that $\operatorname{Int}(A)$ is the largest subset of X that is contained in A and which is open in X. Furthermore, $\operatorname{Int}(A) \subseteq A \subseteq \overline{A}$.

Example 3.30 Let X be a topological space consisting of the set $\{a, b, c\}$ and the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then the closed subsets in X are $\emptyset, \{b, c\}, \{c\}$ and X. Thus the intersection of all of the closed subsets that contain $\{b\}$ is simply $\{b, c\} \cap X = \{b, c\}$, and hence, $\overline{\{b\}} = \{b, c\}$.

Example 3.31 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Assume that $a < b$ are real numbers. Then $\overline{(a, b]} = [a, b]$ and $\operatorname{Int}((a, b]) = (a, b)$.

Let (X, d) be a metric space. If we consider X as a topological space with the metric topology $\mathcal{T}_d$, the closure $\overline{\mathrm{B}(x; r)}$ of an open ball $\mathrm{B}(x; r)$ about $x \in X$ is, in general, *not* the same as the closed ball $\overline{\mathrm{B}}(x; r)$. If d is the discrete metric and X has at least two elements, then $\overline{\mathrm{B}(x; 1)} = \{x\}$ while $\overline{\mathrm{B}}(x; 1) = X$. It is always the case that $\overline{\mathrm{B}(x; r)} \subseteq \overline{\mathrm{B}}(x; r)$.

Definition 3.32 (Dense) Let X be a topological space, and let A be a subset of X. We say that A is *dense in X* if $\overline{A} = X$.

From the definition it follows that A is dense in X if and only if $A \cap U \neq \emptyset$ for every nonempty subset U of X which is open in X.

Example 3.33 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Then the subset $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$: $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 3.34 For any topological space X, the subset X is dense in X. If X is a discrete topological space, then the subset X is the only dense subset in X.

Theorem 3.35 *Let $f\colon X \to Y$ be a map between topological spaces. Then the following are equivalent:*

(1) *f is continuous;*

(2) *for every subset A of X, we have $f(\overline{A}) \subseteq \overline{f(A)}$;*

(3) *for every closed subset B of Y, the preimage $f^{-1}(B)$ of B under f is closed in X.*

Proof. By passing to complements, it follows readily that (1) and (3) are equivalent. We will prove that (2) is equivalent to (3).

Assume (2). Let B be a subset of Y that is closed in Y, and let $A = f^{-1}(B)$. We must show that A is closed. We have $f(A) \subseteq B$. If $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq B$. Hence, $x \in f^{-1}(B) = A$. In other words, $\overline{A} \subseteq A$. Thus $A = \overline{A}$, and hence, $f^{-1}(B)$ is closed in X.

Now assume (3). Let A be a subset of X. We must show that $f(\overline{A}) \subseteq \overline{f(A)}$. Since $\overline{f(A)}$ is closed in Y, it follows by assumption that $f^{-1}(\overline{f(A)})$ is closed in X. Furthermore, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. Since $f^{-1}(\overline{f(A)})$ is closed in X, it follows that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Hence, $f(\overline{A}) \subseteq \overline{f(A)}$. □

3.5 Exercises

Exercise 3.1 Let $X = \{a, b, c, d\}$. Show that $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, X\}$ is *not* a topology on X. Find a topology $\mathcal{T}'$ (different from the discrete topology) on X such that $\mathcal{T} \subseteq \mathcal{T}'$.

Exercise 3.2 Let X be a non-empty set, and let x_0 be an element of X. Show that

$$\mathcal{T} = \{U \subseteq X \mid x_0 \notin U \text{ or } X \setminus U \text{ is finite}\}$$

is a topology on X.

Exercise 3.3 Let X be a set, and let A be a subset of X. Define the coarsest topology on X such that A is open in X.

Exercise 3.4 Show that the discrete topology $\mathcal{T}_{\text{disc}}$ is finer than the cofinite topology $\mathcal{T}_{\text{cof}}$ on any set X.

Exercise 3.5 Let $X = \{a, b, c, d\}$. Find two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ with $\mathcal{T}_1 \neq \mathcal{T}_2$ such that a bijection $f: X \to X$ is a homeomorphism (where the domain is given $\mathcal{T}_1$ as topology and the codomain is given $\mathcal{T}_2$ as topology).

Exercise 3.6 Let X be a topological space, and let A and B be subsets of X.

(a) Assume that $A \subseteq B$. Show that $\overline{A} \subseteq \overline{B}$.

(b) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

We say that a set A *intersects* or *meets* a set B if $A \cap B \neq \emptyset$.

Exercise 3.7 Let X be a topological space, and let A be a subset of X. Show that $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

Exercise 3.8 Let $X = \{a, b, c, d, e\}$, and let

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d, e\}, \{b\}, \{b, c\}, \{b, d, e\}, \{b, c, d, e\}, \{d, e\}, X\}$$

be a topology on X. Is the subset $\{a, b\}$ dense in X?

4. Generating topologies

4.1 Generating topologies from subsets

The following theorem tells us how we may extract a third topology from two other topologies on the same set.

Theorem 4.1 **(The intersection of two topologies is a topology)** *Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on X.*

Proof. Clearly, $\emptyset$ and X are in $\mathcal{T}_1 \cap \mathcal{T}_2$, so T1 is satisfied.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of sets such that $U_\lambda \in \mathcal{T}_1 \cap \mathcal{T}_2$ for each $\lambda \in \Lambda$ where Λ is some index set. Then, for $i = 1, 2$, $U_\lambda \in \mathcal{T}_i$ for each $\lambda \in \Lambda$. Thus $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_i$ for $i = 1, 2$. Hence, $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_1 \cap \mathcal{T}_2$, and so, T2 is satisfied.

Finally, to prove that T3 is satisfied, let $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$. Thus, for $i = 1, 2$, $U, V \in \mathcal{T}_i$ implies that $U \cap V \in \mathcal{T}_i$. Hence, $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$. □

Theorem 4.1 may be extended to hold for a family of topologies: if $\{\mathcal{T}_\lambda\}_{\lambda \in \Lambda}$ is a family of topologies on X, then $\bigcap_{\lambda \in \Lambda} \mathcal{T}_\lambda$ is also a topology on X. If we follow the convention that for subsets S of a fixed (large) set U,

$$\bigcap_{S \in \emptyset} S = U,$$

then the extended version of Theorem 4.1 may also include an empty family $\{\mathcal{T}_\lambda\}_{\lambda \in \emptyset}$ of topologies with

$$\bigcap_{\lambda \in \emptyset} \mathcal{T}_\lambda = \mathcal{P}(X),$$

i.e., the discrete topology on X (with our fixed (large) set U being equal to $\mathcal{P}(X)$). However, not all mathematicians follow this convention. Thus we will in general not define the intersection of an empty family.

The union of two topologies is not necessarily a topology.

Example 4.2 Let $X = \{a, b, c\}$, and let $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{c\}, X\}$ be two topologies on X. Then

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, X\}$$

is the indiscrete topology on X while

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, \{c\}, X\}$$

is *not* a topology on X: $\mathcal{T}_1 \cup \mathcal{T}_2$ does not satisfy T2.

Recall from Section 3.1 that for any set X the discrete topology $\mathcal{T}_{\text{disc}}$ is the largest topology we may equip X with, and the indiscrete topology $\mathcal{T}_{\text{ind}}$ is the smallest topology we may equip X with. For any topology $\mathcal{T}$ on X we have

$$\mathcal{T}_{\text{ind}} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\text{disc}}.$$

That is, we have partially ordered topologies on X by inclusion.

Let X be a set. We often want to have a collection of subsets $\mathcal{S}$ of X to be the open subsets of a topology on X.

Definition 4.3 (Topology generated by a collection of subsets) Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X. The *topology generated by* $\mathcal{S}$ is the topology

$$\langle \mathcal{S} \rangle = \bigcap_{\substack{\mathcal{T} \text{ topology} \\ \mathcal{S} \subseteq \mathcal{T}}} \mathcal{T}$$

on X.

In other words, $\langle \mathcal{S} \rangle$ contains $\mathcal{S}$ and for any other topology $\mathcal{T}'$ containing $\mathcal{S}$, we have $\langle \mathcal{S} \rangle \subseteq \mathcal{T}'$. Thus $\langle \mathcal{S} \rangle$ is unique.

Example 4.4 Let X be a set, and let $\mathcal{S} = \emptyset$. Then $\langle \mathcal{S} \rangle$ is the same as the indiscrete topology on X, i.e.,

$$\langle \mathcal{S} \rangle = \mathcal{T}_{\text{ind}} = \{\emptyset, X\}.$$

Example 4.5 Let X be a set, and let $\mathcal{S}$ be the collection of all the singletons of X, i.e., $\mathcal{S} = \{\{x\} \mid x \in X\}$. Then $\langle \mathcal{S} \rangle$ is the same as the discrete topology on X, i.e.,

$$\langle \mathcal{S} \rangle = \mathcal{T}_{\text{disc}} = \mathcal{P}(X).$$

4.2 Basis for a topology

It is often convenient to define a topology $\mathcal{T}$ on a set X by only specifying a subcollection $\mathcal{B}$ of $\mathcal{T}$ satisfying certain properties. The open subsets of X are then precisely the unions of subcollections of $\mathcal{B}$. In this way, we say the basis determines, or generates, the topology.

Definition 4.6 (Basis) Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

B1 for each $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$;

B2 if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The elements of $\mathcal{B}$ are sometimes referred to as *basis elements*. Basis elements are subsets of X.

Example 4.7 Let X be a set, and let $\mathcal{B}$ be the collection of all the singletons of X. Then $\mathcal{B}$ is a basis for the discrete topology on X.

Example 4.8 Let (X, d) be a metric space. Then the collection of (open) ϵ-balls

$$\mathcal{B} = \{\mathrm{B}(x; \epsilon) \mid x \in X, \epsilon > 0\}$$

is a basis for the metric topology $\mathcal{T}_d$, as defined in Theorem 3.2, on X.

The following theorem describes a topology generated by a basis.

Theorem 4.9 *Let X be a set, and let $\mathcal{B}$ be basis for a topology on X. The collection $\mathcal{T}$ generated by $\mathcal{B}$ of subsets U of X with the property that for each $x \in U$ there is a basis element $B \in \mathcal{B}$ with $x \in B \subseteq U$ is a topology on X.*

Proof. Clearly, $\emptyset$ and X are both in $\mathcal{T}$. Hence, T1 is satisfied.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a subcollection of $\mathcal{T}$. Let $V = \bigcup_{\lambda \in \Lambda} U_\lambda$. We need to prove that $V \in \mathcal{T}$. Let $x \in V$. Then there is a $\lambda \in \Lambda$ such that $x \in U_\lambda$. Since $U_\lambda \in \mathcal{T}$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U_\lambda$. As $U_\lambda \subseteq V$, it follows that $x \in B \subseteq V$. Hence, $V \in \mathcal{T}$, and so, T2 is satisfied.

Let $U_1, U_2 \in \mathcal{T}$. We need to prove that $U_1 \cap U_2 \in \mathcal{T}$. Let $x \in U_1 \cap U_2$. Since $U_1 \cap U_2 \subseteq U_i$ we have $x \in U_i$, and thus there is a basis element $B_i \in \mathcal{B}$ with $x \in B_i \subseteq U_i$ for $i = 1, 2$. Hence, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$. By B2 there is a basis element $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. Thus $x \in B_3 \subseteq U_1 \cap U_2$, and hence, T3 is satisfied. □

The topology generated by a basis may also be described using the following theorem.

Theorem 4.10 *Let X be a set, and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $B \in \mathcal{B}$ be any basis element. Then for each $x \in B$ we obviously have $x \in B$ and $B \subseteq B$. Thus $B \in \mathcal{T}$. It follows that any union of basis elements is a union of elements of $\mathcal{T}$, and hence, is in $\mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. For each $x \in U$ there is a $B_x \in \mathcal{B}$ with $x \in B_x$ and $B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$, and thus, U is the union of elements of $\mathcal{B}$. □

We end this section with a theorem describing a criterion for whether one topology is finer than another when both topologies are described using bases.

Theorem 4.11 *Let X be a set, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, on X. Then the following are equivalent:*

(1) *$\mathcal{T}_2$ is finer than $\mathcal{T}_1$, i.e., $\mathcal{T}_1 \subseteq \mathcal{T}_2$.*

(2) *For each $B_1 \in \mathcal{B}_1$ and each $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

Proof. Assume (1). Let $B_1 \in \mathcal{B}_1$ and $x \in B_1$. Since $B_1 \in \mathcal{T}_1$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we have $B_1 \in \mathcal{T}_2$. Furthermore, as $\mathcal{T}_2$ is the topology generated by $\mathcal{B}_2$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2$ where $B_2 \subseteq B_1$. Hence, (2) is satisfied.

Now assume (2). Let $U \in \mathcal{T}_1$. We must prove that $U \in \mathcal{T}_2$. Since $\mathcal{B}_1$ generates $\mathcal{T}_1$, then for each $x \in U$ there is a $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq U$. By assumption there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Hence, $B_2 \subseteq U$, and so, $U \in \mathcal{T}_2$. Thus (1) is satisfied. □

In order to have $\mathcal{T}_1 \subseteq \mathcal{T}_2$ it is *not* necessary to have $\mathcal{B}_1 \subseteq \mathcal{B}_2$, i.e., each basis element in $\mathcal{B}_1$ need not be a basis element in $\mathcal{B}_2$. However, for each basis element $B_1 \in \mathcal{B}_1$ and each point $x \in B_1$ there should be some (possibly) smaller basis element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

4.3 Subbasis for a topology

Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X. We can form a basis $\mathcal{B}$ for a topology by simply taking all finite intersections

$$B = \bigcap_{i=1}^{n} S_i$$

of elements of $\mathcal{S}$. Thus the open sets in the topology generated by this basis are all unions of such basis elements B, cf. Theorem 4.10. Thus the open sets are all unions of all finite intersections of elements of $\mathcal{S}$. The collection $\mathcal{S}$ is then referred to as a subbasis.

Definition 4.12 (Subbasis) Let X be a set. A *subbasis* for a topology on X is a collection $\mathcal{S}$ of subsets of X whose union equals X.

Lemma 4.13 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. The collection $\mathcal{B}$ consisting of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X and is called the basis associated to $\mathcal{S}$.*

Proof. Each $x \in X$ must lie in some $S \in \mathcal{S}$. Hence, $x \in S$. Thus x is an element of the basis element S in $\mathcal{B}$, and so, B1 is satisfied.

Let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{i=1}^{n} S_i'$ be two basis elements of $\mathcal{B}$, and let $x \in B_1 \cap B_2$. We must prove that there is a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Let

$$B_3 = \left(\bigcap_{i=1}^{m} S_i\right) \cap \left(\bigcap_{i=1}^{n} S_i'\right).$$

Then B_3 is also a finite intersection of elements of $\mathcal{S}$, and hence, $B_3 \in \mathcal{B}$ with $x \in B_3$. Thus B2 is satisfied. □

By combining the previous lemma with Theorem 4.10, we get the following lemma.

Lemma 4.14 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. The collection $\mathcal{T}$ generated by $\mathcal{S}$ consisting of all unions of all basis elements of the associated basis $\mathcal{B}$ is a topology on X.*

When referring to the topology $\mathcal{T}$ generated by the subbasis $\mathcal{S}$ we mean the topology generated by the associated basis $\mathcal{B}$. We have $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{T}$.

The following theorem provides an explicit description of the topology generated by a collection of subsets $\mathcal{S}$ of a set X.

Theorem 4.15 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. Then there is a unique topology $\langle\mathcal{S}\rangle$ generated by $\mathcal{S}$ which is coarser than any other topology containing $\mathcal{S}$, where*

$$\langle\mathcal{S}\rangle = \left\{ \bigcup_{\lambda\in\Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \,\middle|\, S_{\lambda,i} \in \mathcal{S} \right\}.$$

In other words, $\langle\mathcal{S}\rangle$ is the topology for which $\mathcal{S}$ is a subbasis.

Proof. Since the discrete topology $\mathcal{T}_{\text{disc}} = \mathcal{P}(X)$, there is at least one topology on X that contains $\mathcal{S}$. We know from Theorem 4.1 that taking the intersection of all topologies that contain $\mathcal{S}$ is again a topology which clearly still contains $\mathcal{S}$. By construction, $\langle\mathcal{S}\rangle$ is then contained in any other topology containing $\mathcal{S}$. Thus $\langle\mathcal{S}\rangle$ is the unique topology with this property.

Let

$$\mathcal{T}_{\mathcal{S}} = \left\{ \bigcup_{\lambda\in\Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \,\middle|\, S_{\lambda,i} \in \mathcal{S} \right\}.$$

Clearly, $\mathcal{T}_{\mathcal{S}} \subseteq \langle\mathcal{S}\rangle$. We need to prove that they are equal. To do this we will prove that $\mathcal{T}_{\mathcal{S}}$ is a topology on X that contains $\mathcal{S}$. Hence, by the first part $\langle\mathcal{S}\rangle = \mathcal{T}_{\mathcal{S}}$. Since $\mathcal{S}$ is a subbasis for a topology on X, by Lemma 4.14 we know that the topology generated by $\mathcal{S}$ is equal to the collection of all unions of basis elements of the associated $\mathcal{B}$ to $\mathcal{S}$. Hence, $\mathcal{T}_{\mathcal{S}}$ is a topology on X. □

We end this section with a theorem about continuity and (sub)basis.

Theorem 4.16 *Let X and Y be topological spaces, and let $\mathcal{B}$ (resp., $\mathcal{S}$) be a basis (resp., subbasis) for the topology on Y. Then a map $f : X \to Y$ is continuous if and only if for each $B \in \mathcal{B}$ (resp. $S \in \mathcal{S}$) the preimage $f^{-1}(B)$ (resp., $f^{-1}(S)$) is open in X.*

Proof. We prove the statement about basis.

Assume that f is continuous. Since each basis element $B \in \mathcal{B}$ is open in Y, then by continuity $f^{-1}(B)$ is open in X.

Assume that for each $B \in \mathcal{B}$ the preimage $f^{-1}(B)$ is open in X. Let $\mathcal{T}_Y$ be the topology on Y. Since every $V \in \mathcal{T}_Y$ is a union $V = \bigcup_{\lambda\in\Lambda} B_\lambda$ of basis elements $B_\lambda \in \mathcal{B}$, we have

$$f^{-1}(V) = \bigcup_{\lambda\in\Lambda} f^{-1}(B_\lambda).$$

Thus if each $f^{-1}(B_\lambda)$ is open in X, so is $f^{-1}(V)$. □

4.4 Exercises

Exercise 4.1 Let $X = \{a, b, c, d, e\}$, and let

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, d, e\}, \{b\}, \{b, d, e\}, \{b, c, d, e\}, \{c, d, e\}, \{d, e\}, X\}$$

be a topology on X. Show that $\mathcal{S} = \{\{a, b\}, \{b, d, e\}, \{c, d, e\}\}$ is a subbasis for $\mathcal{T}$. Is $\mathcal{S}' = \{\{a, b\}, \{b, c, d, e\}, \{d, e\}\}$ a subbasis for $\mathcal{T}$?

Exercise 4.2 Let $\mathcal{B}$ be the collection of all open intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *standard topology* on $\mathbb{R}$ denoted by $\mathcal{T}_{\text{std}}$.

(b) Show that $\mathcal{T}_{\text{std}} = \mathcal{T}_d$ where $\mathcal{T}_d$ is the metric topology obtained from the metric $d(x, y) = |x - y|$.

Exercise 4.3 Show that

$$\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$$

is a subbasis for the standard topology on $\mathbb{R}$.

Exercise 4.4 Let $\mathbb{Q}$ deonte the set of rational numbers, and let $\mathbb{R}$ denote the set of real numbers. Show that

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}$ where $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

Exercise 4.5 Let $\mathcal{B}$ be the collection of all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *lower limit topology* on $\mathbb{R}$.

(b) Find the closure of the subset $(0, 1)$ of $\mathbb{R}$ given the lower limit topology.

Exercise 4.6 For each $n \in \mathbb{Z}$, let

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

Show that the collection $\mathcal{B} = \{B(n) \mid n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$. The topology generated by $\mathcal{B}$ is known as the *digital line topology* on $\mathbb{Z}$. See [1, pp. 44–46] for some applications of this topology.

Exercise 4.7 Let $\mathcal{B}$ be the collection of all subsets of the form $A_{a,b} = \{az + b \mid z \in \mathbb{Z}\}$ of $\mathbb{Z}$, where $a, b \in \mathbb{Z}$ and $a \neq 0$. (The set $A_{a,b}$ is known as an *arithmetic progression*.)

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

(b) Show that there are infinitely many primes by using the topology generated by $\mathcal{B}$. (This topology is known as the *arithmetic progression topology* on $\mathbb{Z}$ and it was used originally by Furstenberg [3] to show that there are infinitely many primes.)

Exercise 4.8 Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Show that a subset A of X is dense in X if and only if every non-empty basis element in $\mathcal{B}$ intersects A. (Recall that a set U *intersects* a set V if $U \cap V \neq \emptyset$.)

5. Constructing topological spaces

5.1 Subspaces

Let A be a subset of a topological space X. There is a natural way to define a topology on A that is based on the topology on X.

Definition 5.1 (Subspace topology) Let X be a topological space, and let A be a subset of X. The collection

$$\mathcal{T}_A = \{A \cap U \mid U \text{ is open in } X\}$$

of subsets of A is called the *subspace topology* on A.

The subspace topology is indeed a topology.

Lemma 5.2 *Let X be a topological space, and let A be a subset of X. Then the collection $\mathcal{T}_A = \{A \cap U \mid U \text{ is open in } X\}$ is a topology on A.*

Proof. Let $\mathcal{T}$ denote the topology on X.

Since $\emptyset, X \in \mathcal{T}$, $\emptyset = A \cap \emptyset$ and $A = A \cap X$, then, clearly, $\emptyset, A \in \mathcal{T}_A$. Hence, T1 is satisfied.

Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of A who are open in A, i.e., $V_\lambda \in \mathcal{T}_A$. We must show that $\bigcup_{\lambda \in \Lambda} V_\lambda \in \mathcal{T}_A$. For each $\lambda \in \Lambda$ there is a $U_\lambda \in \mathcal{T}$ such that $V_\lambda = A \cap U_\lambda$. Thus

$$\bigcup_{\lambda \in \Lambda} V_\lambda = \bigcup_{\lambda \in \Lambda} (A \cap U_\lambda) = A \cap \bigcup_{\lambda \in \Lambda} U_\lambda.$$

Since $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$, it follows that $\bigcup_{\lambda \in \Lambda} V_\lambda \in \mathcal{T}_A$. Hence, T2 is satisfied.

Let $V_1, V_2, \ldots, V_n$ be subsets of A that are open in A, i.e., $V_i \in \mathcal{T}_A$ for $i = 1, 2, \ldots, n$. We must show that $\bigcap_{i=1}^n V_i \in \mathcal{T}_A$. For each $i \in \{1, 2, \ldots, n\}$ there is a $U_i \in \mathcal{T}$ such that $V_i = A \cap U_i$. Thus

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (A \cap U_i) = A \cap \bigcap_{i=1}^n U_i.$$

Since $\bigcap_{i=1}^n U_i \in \mathcal{T}$, it follows that $\bigcap_{i=1}^n V_i \in \mathcal{T}_A$. Hence, T3 is satisfied. □

Let X be a topological space (with topology $\mathcal{T}$), and let A be a subspace of X. If V is a subset of A, there are two possible meanings to the statement "V is open." We can either take V to be open in X, i.e., $V \in \mathcal{T}$, or we can take V to be open in A, i.e., $V \in \mathcal{T}_A$. In general, these do *not* mean the same thing.

We say that V is *closed in A* if V is closed in the subspace topology of A. In other words, if V is closed in A, then $A \setminus V$ is open in A.

Theorem 5.3 *Let X be a topological space, and let A be a subspace of X. Then a subset K of A is closed in A if and only if there is a closed subset L of X with $K = A \cap L$.*

Proof. Assume that K is a closed subset of A. Then $V = A \setminus K$ is open in A, i.e., there is an open subset U of X with $V = A \cap U$. Moreover, $L = X \setminus U$ is a closed subset of X and

$$A \cap L = A \cap (X \setminus U) = A \setminus (A \cap U) = A \setminus V = K.$$

See Figure 5.1.

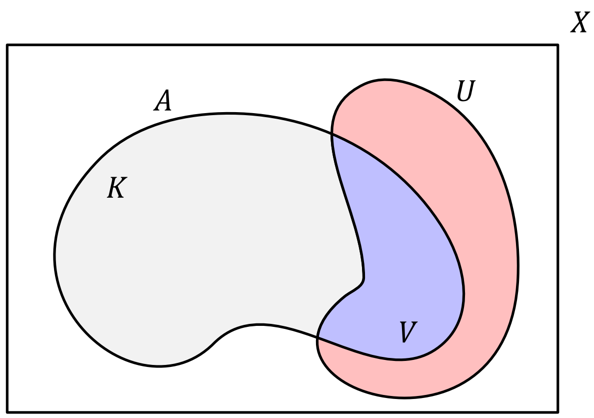


Figure 5.1: $K = A \cap L$

Now assume that L is a closed subset of X with $K = A \cap L$. Then $U = X \setminus L$ is an open subset of X, and so, $V = A \cap U$ is an open subset of A. Furthermore,

$$A \setminus K = A \setminus (A \cap L) = A \cap (X \setminus L) = A \cap U = V.$$

See Figure 5.2. Hence, K is a closed subset of A. □

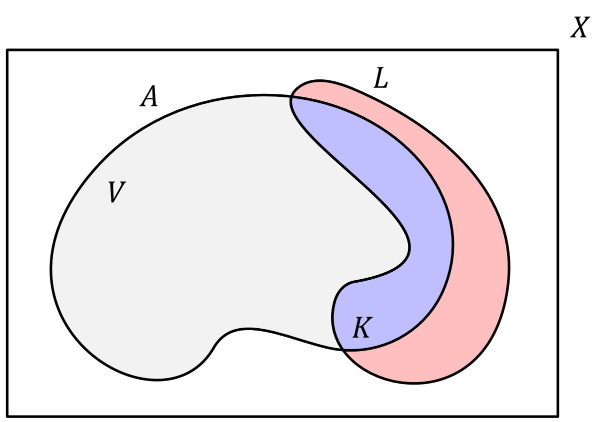


Figure 5.2: $V = A \setminus K$

The next example illustrates the fact that we may have subsets of a topological space X that are open in the subspace A but which are not open in X.

Example 5.4 Let $\mathbb{R}$ denote the set of real numbers equipped with the standard topology, cf. Exercise 4.2, and let $I = [0, 1]$ be a subspace of $\mathbb{R}$. Then sets of the form $[0, a)$ and $(a, 1]$ with $0 < a < 1$ are open in I but not in $\mathbb{R}$.

The following theorem describes how we may extract a basis for the subspace topology on A from the basis of a topology on X.

Theorem 5.5 *Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. If A is a subset of X, then the collection*

$$\mathcal{B}_A = \{A \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on A.

Proof. We need to prove that $\mathcal{B}_A$ is a basis for the subspace topology on A. We first prove that $\mathcal{B}_A$ is a basis for a topology on A, and then that the topology generated by $\mathcal{B}_A$ equals the subspace topology on A.

First note that each $B \in \mathcal{B}$ is open in X, and so, each $A \cap B \in \mathcal{B}_A$ is open in A. For each $x \in X$ there is a basis element $B \in \mathcal{B}$ such that $x \in B$. Let $x \in A$. Since $A = A \cap X$, there must be a basis element $B \in \mathcal{B}$ such that $x \in A \cap B$. Hence, B1 holds. Now let $x \in A \cap (B_1 \cap B_2)$ where B_1 and B_2 are basis elements of $\mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on X, it follows that there is a basis element B_3 of $\mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$ and $x \in A \cap B_3 \subseteq A \cap (B_1 \cap B_2)$. Hence, B2 is satisfied. Thus $\mathcal{B}_A$ is a basis for a topology on A.

Let $\mathcal{T}_A$ be the subspace topology on A. We want to prove that the topology $\mathcal{T}'$ generated by $\mathcal{B}_A$ is equal to $\mathcal{T}_A$. If $A \cap U \in \mathcal{T}_A$ and $x \in A \cap U$, then, using the fact that $\mathcal{B}$ is a basis for the topology on X, we have $B \in \mathcal{B}$ such that $x \in A \cap B \subseteq A \cap U$. Thus $A \cap U \in \mathcal{T}'$, cf. Theorem 4.9. By Theorem 4.10 we know that $\mathcal{T}'$ is equal to the collection of all unions of elements of $\mathcal{B}_A$. Hence, if $W \in \mathcal{T}'$ then W equals a union of elements of $\mathcal{B}_A$. Since each element of $\mathcal{B}_A$ belongs to $\mathcal{T}_A$ and $\mathcal{T}_A$ is a topology, W also belongs to $\mathcal{T}_A$. □

We end this section with an alternative description of the subspace topology. Let X be a topological space, and let T be a set. There do exist topologies on T that make $f : T \to X$ continuous, e.g., the discrete topology. If $\mathcal{T}_f$ is the intersection of all topologies on T such that f is continuous, then $\mathcal{T}_f$ is the coarsest topology for which f is continuous and

$$\mathcal{T}_f = \{f^{-1}(U) \mid U \text{ is open in } X\}.$$

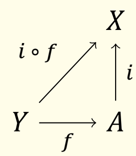
From this we may define the subspace topology as follows: Let X be a topological space, and let A be subset of X. The *subspace topology* on A is then the coarsest topology on A for which the inclusion $i : A \to X$, given by $i(x) = x$ for $x \in A$, is continuous. This coincides with our previous definition as $i^{-1}(U) = A \cap U$ for any subset U of X. Thus

$$\mathcal{T}_i = \{i^{-1}(U) \mid U \text{ is open in } X\} = \{A \cap U \mid U \text{ is open in } X\},$$

and hence, $\mathcal{T}_i = \mathcal{T}_A$.

The following theorem describes a universal property for the subspace topology.

Theorem 5.6 *Let X be a topological space, and let A be a subset of X. Then the subspace topology on A is the only topology on A with the following universal property: for every topological space Y and every map $f: Y \to A$, f is continuous if and only if $i \circ f: Y \to X$ is continuous where $i: A \to X$ is the inclusion map given by $i(x) = x$ for $x \in A$.*



Proof. We will first prove that the subspace topology $\mathcal{T}_A$ has the universal property that for every topological space Y and every map $f: Y \to A$, f is continuous if and only if $i \circ f: Y \to X$ is continuous. Then we will prove that $\mathcal{T}_A$ is the only topology on A with this property.

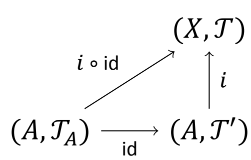
Consider A as a subspace of X. Assume that f is continuous. Since the inclusion map i is continuous (with A given the subspace topology), and the composition of two continuous maps is again continuous, cf. Theorem 3.16, it follows that $i \circ f$ is continuous. Now assume that $i \circ f$ is continuous. Let V be an open set in A, i.e., $V = A \cap U$ for some open set U in X. Since

$$f^{-1}(V) = f^{-1}(A \cap U) = f^{-1}(i^{-1}(U)) = (i \circ f)^{-1}(U)$$

is open in Y by continuity of $i \circ f$, it follows that f is continuous. Thus the subspace topology $\mathcal{T}_A$ has the desired property.

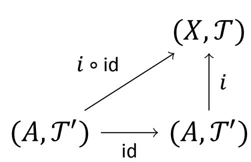
Let $\mathcal{T}'$ be a topology on A with the universal property that for every topological space Y and every map $f: Y \to A$, f is continuous if and only if $i \circ f: Y \to X$ is continuous. We must show that $\mathcal{T}_A = \mathcal{T}'$.

Let $\mathcal{T}$ be the topology on X, and let A be given the topology $\mathcal{T}'$. First let $Y = A$ with the subspace topology. Then for $f = \mathrm{id}: (A, \mathcal{T}_A) \to (A, \mathcal{T}')$, we have $i \circ \mathrm{id} = i: (A, \mathcal{T}_A) \to (X, \mathcal{T})$ which is continuous. Hence, by the universal property id is continuous.



Thus any $V \in \mathcal{T}'$ must also be an element of $\mathcal{T}_A$, and so, $\mathcal{T}' \subseteq \mathcal{T}_A$.

Secondly let $Y = A$ with $\mathcal{T}'$ as its topology. Then, clearly, $f = \mathrm{id}: (A, \mathcal{T}') \to (A, \mathcal{T}')$ is continuous. Thus by the universal property it follows that $i \circ \mathrm{id} = i: (A, \mathcal{T}') \to (X, \mathcal{T})$ is continuous.



Thus for any $U \in \mathcal{T}$, we have $i^{-1}(U) = A \cap U \in \mathcal{T}'$. That is, $\mathcal{T}_A \subseteq \mathcal{T}'$. Hence, $\mathcal{T}_A = \mathcal{T}'$. □

5.2 Products

Let X and Y be topological spaces. If we want to give the product $X \times Y$ a topology, a first approach might be to take the collection

$$\mathcal{C} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

as the topology on $X \times Y$. However, $\mathcal{C}$ is *not* a topology. The union of two elements of $\mathcal{C}$ is not necessarily of the form $U \times V$ for some U open in X and some V open in Y. See Figure 5.3.

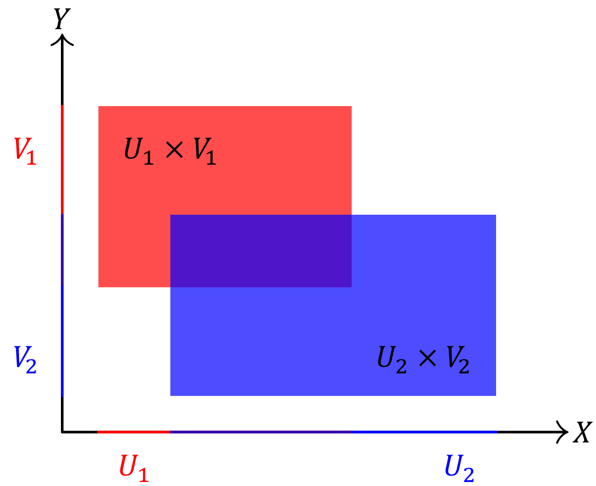


Figure 5.3: The collection $\mathcal{C}$ of all products of open sets in X and in Y is not a topology on $X \times Y$.

We can remedy the situation by taking $\mathcal{C}$ as a basis instead. The topology generated from this basis is what we will take to be the product topology on $X \times Y$.

Definition 5.7 **(Product topology)** Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

Lemma 5.8 *Let X and Y be topological spaces. Then the collection*

$$\mathcal{B} = \{U \times V \mid U \textit{ is open in } X \textit{ and } V \textit{ is open in } Y\}$$

is a basis for a topology on $X \times Y$.

Proof. Let $(x, y) \in X \times Y$. We need to show that there is a basis element $U \times V \in \mathcal{B}$ such that $(x, y) \in U \times V \subseteq X \times Y$. Since X is open in X and Y is open in Y, we simply take $U = X$ and $V = Y$. Thus B1 is satisfied.

Now let $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ where $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$. Since

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

and $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively, it follows by letting $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$ that there is a basis element $U_3 \times V_3 \in \mathcal{B}$ such that $(x, y) \in U_3 \times V_3 \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$. Hence, B2 holds. Thus $\mathcal{B}$ is a basis for a topology on $X \times Y$. □

The basis described in Lemma 5.8 is relatively large; it consists of pairs of every open set U in X and every open set V in Y. The following theorem describes a smaller basis for the product topology based on bases rather than whole topologies.

Theorem 5.9 *Let X and Y be topological spaces. If $\mathcal{B}_X$ is a basis for the topology on X and $\mathcal{B}_Y$ is a basis for the topology on Y, then the collection*

$$\mathcal{B}_{X\times Y} = \{B_X \times B_Y \mid B_X \in \mathcal{B}_X \text{ and } B_Y \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$.

Proof. We follow the arguments for the proof of Theorem 5.5 and adapt them to our current setting.

First note that each $B_X \times B_Y \in \mathcal{B}_X \times \mathcal{B}_Y$ is open in $X \times Y$ as each B_X is open in X and each B_Y is open in Y. Let $(x, y) \in X \times Y$. Using the fact that $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for topologies on X and Y, respectively, there are basis elements $B_X \in \mathcal{B}_X$ and $B_Y \in \mathcal{B}_Y$ such that $(x, y) \in B_X \times B_Y \subseteq X \times Y$. Thus B1 is satisfied. Now let $(x, y) \in (B_{X,1} \times B_{Y,1}) \cap (B_{X,2} \times B_{Y,2})$ where $B_{X,1}, B_{X,2} \in \mathcal{B}_X$ and $B_{Y,1}, B_{Y,2} \in \mathcal{B}_Y$. Since

$$(B_{X,1} \times B_{Y,1}) \cap (B_{X,2} \times B_{Y,2}) = (B_{X,1} \cap B_{X,2}) \times (B_{Y,1} \cap B_{Y,2})$$

and $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for the topologies on X and Y, respectively, it follows that there are basis elements $B_{X,3} \in \mathcal{B}_X$ and $B_{Y,3} \in \mathcal{B}_Y$ such that $x \in B_{X,3} \subseteq B_{X,1} \cap B_{X,2}$, $y \in B_{Y,3} \subseteq B_{Y,1} \cap B_{Y,2}$ and $(x, y) \in B_{X,3} \times B_{Y,3} \subseteq (B_{X,1} \cap B_{X,2}) \times (B_{Y,1} \cap B_{Y,2})$. Thus B2 holds. Hence, $\mathcal{B}_{X\times Y}$ is a basis for a topology on $X \times Y$.

Let $\mathcal{T}_{X\times Y}$ be the product topology on $X \times Y$. We want to prove that the topology $\mathcal{T}'$ generated by $\mathcal{B}_{X\times Y}$ is equal to $\mathcal{T}_{X\times Y}$. Let $W \in \mathcal{T}_{X\times Y}$, and let $(x, y) \in W$. Then there is an open set U in X and an open set V in Y such that $(x, y) \in U \times V \subseteq W$, cf. Theorem 4.9. Since U is open in X and $\mathcal{B}_X$ is a basis for the topology on X, it follows that there is a basis element $B_X \in \mathcal{B}_X$ such that $x \in B_x \subseteq U$. Likewise, there is a basis element $B_Y \in \mathcal{B}_Y$ such that $y \in B_Y \subseteq V$. Thus $(x, y) \in B_X \times B_Y \subseteq W$, and so, $W \in \mathcal{T}'$, cf. Theorem 4.9. By Theorem 4.10, we know that $\mathcal{T}'$ is equal to the collection of all unions of elements of $\mathcal{B}_{X\times Y}$. Hence, if $W \in \mathcal{T}'$ then W equals a union of elements of $\mathcal{B}_{X\times Y}$. Since each element of $\mathcal{B}_{X\times Y}$ belongs to $\mathcal{T}_{X\times Y}$ and $\mathcal{T}_{X\times Y}$ is a topology, W also belongs to $\mathcal{T}_{X\times Y}$. □

Example 5.10 Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3\}$ with topologies $\mathcal{T}_X = \{\emptyset, \{a, b\}, \{b\}, \{b, c, d, e\}, X\}$ and $\mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 2\}, Y\}$, respectively. Then the collection

$$\begin{aligned}\mathcal{B}_{X\times Y} = \big\{&\{a, b\} \times \{1\}, \{b\} \times \{1\}, \{b, c, d, e\} \times \{1\},\\ &\{a, b\} \times \{1, 2\}, \{b\} \times \{1, 2\}, \{b, c, d, e\} \times \{1, 2\},\\ &\{a, b\} \times Y, \{b\} \times Y, \{b, c, d, e\} \times Y\big\}\end{aligned}$$

is a basis for the product topology on $X \times Y$.

Example 5.11 Let $\mathbb{R}$ denote the set of real numbers equipped with the standard topology, cf. Exercise 4.2. Then the collection

$$\mathcal{B}_{\mathbb{R}^2} = \{(a, b) \times (c, d) \mid a < b, c < d\}$$

of open rectangular regions in $\mathbb{R}^2$ is a basis for the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, since a basis for the standard topology of $\mathbb{R}$ is the collection of open intervals of the form (a, b) where $a < b$.

Example 5.12 Let S^1 denote the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ seen as a subspace of $\mathbb{R}^2$ (with the standard topology), i.e., S^1 is given the subspace topology. A basis element B for the product topology on the torus $T^2 = S^1 \times S^1$ is illustrated in Figure 5.4. Note that the surface depicted in Figure 5.4 is *homeomorphic* to T^2; it is the surface of revolution generated by revolving a circle, say of radius 1 in the xz-plane with center $(2, 0, 0)$, about an axis, e.g., the z-axis.

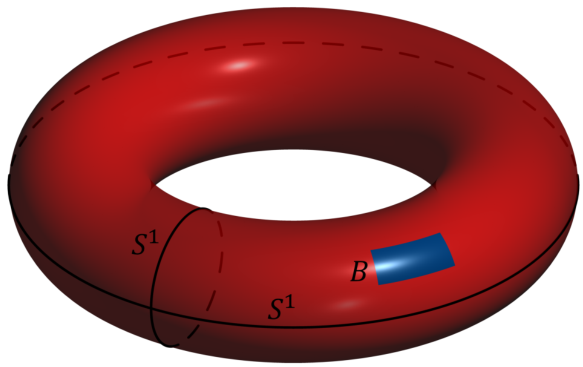


Figure 5.4: An illustration of a basis element B for the product topology on $T^2 = S^1 \times S^1$.

We end this section with an alternative description of the product topology. The map $\pi_1 : X \times Y \to X$ given by

$$\pi_1(x, y) = x$$

for $(x, y) \in X \times Y$ is called the projection of $X \times Y$ onto X. Similarly, the map $\pi_2 : X \times Y \to Y$ given by

$$\pi_2(x, y) = y$$

for $(x, y) \in X \times Y$ is called the projection of $X \times Y$ onto Y.

Theorem 5.13 *Let X and Y be topological spaces. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections of $X \times Y$ onto its first and second factors, respectively. The product topology is the only topology on $X \times Y$ with the following universal property: for every topological space Z and every map $f : Z \to X \times Y$, f is continuous if and only if $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continuous.*

Proof. We follow the arguments for the proof of Theorem 5.6 and adapt them to our current setting.

We first prove that the product topology $\mathcal{T}_{X \times Y}$ has the universal property that for every topological space Z and every map $f : Z \to X \times Y$, f is continuous if and only if $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continuous.

Let $X \times Y$ be given the product topology, and let $f : Z \to X \times Y$ be continuous. Since $\pi_1^{-1}(U) = U \times Y$ for an open set U in X and Y is open in Y, it follows that π_1 is continuous. Likewise, $\pi_2 : X \times Y \to Y$

is continuous. Thus by Theorem 3.16 both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous. Now assume that $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous. Let $U \times V$ be a subset of $X \times Y$ where U is an open set in X and V is an open set in Y. Since $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ and

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(\pi_1^{-1}(U) \cap \pi_2^{-1}(V)) \\ &= f^{-1}(\pi_1^{-1}(U)) \cap f^{-1}(\pi_2^{-1}(V)) \\ &= (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V) \end{aligned}$$

is open in Z by continuity of $\pi_1 \circ f$ and $\pi_2 \circ f$, and subsets of the form $U \times V$ form a basis for the product topology on $X \times Y$, cf. Lemma 5.8, it follows that f is continuous.

Let $\mathcal{T}$ be a topology on $X \times Y$ with the universal property that for every topological space Z and every map $f\colon Z \to X \times Y$, f is continuous if and only if $\pi_1 \circ f\colon Z \to X$ and $\pi_2 \circ f\colon Z \to Y$ are continuous. We must show that $\mathcal{T}_{X\times Y} = \mathcal{T}$.

Let $\mathcal{T}_X$ be the topology on X and $\mathcal{T}_Y$ be the topology on Y, and let $X \times Y$ be given the topology $\mathcal{T}$. First let $Z = X \times Y$ with the product topology. Then for $f = \text{id}\colon (X \times Y, \mathcal{T}_{X\times Y}) \to (X \times Y, \mathcal{T})$, we have $\pi_1 \circ \text{id} = \pi_1\colon (X \times Y, \mathcal{T}_{X\times Y}) \to (X, \mathcal{T}_X)$ and $\pi_2 \circ \text{id} = \pi_2\colon (X \times Y, \mathcal{T}_{X\times Y}) \to (Y, \mathcal{T}_Y)$ which are both continuous. Thus by the universal property id is continuous.

$$\begin{array}{ccc} & & (X \times Y, \mathcal{T}) \\ & \overset{\text{id}}{\nearrow} & \downarrow \pi_1 \\ (X \times Y, \mathcal{T}_{X\times Y}) & \xrightarrow[\pi_1 \circ \text{id}]{} & (X, \mathcal{T}_X) \end{array} \qquad \begin{array}{ccc} & & (X \times Y, \mathcal{T}) \\ & \overset{\text{id}}{\nearrow} & \downarrow \pi_2 \\ (X \times Y, \mathcal{T}_{X\times Y}) & \xrightarrow[\pi_2 \circ \text{id}]{} & (Y, \mathcal{T}_Y) \end{array}$$

Hence, any $W \in \mathcal{T}$ must also be an element of $\mathcal{T}_{X\times Y}$, and so, $\mathcal{T} \subseteq \mathcal{T}_{X\times Y}$.

Secondly let $Z = X \times Y$ with $\mathcal{T}$ as its topology. Then, clearly, $f = \text{id}\colon (X \times Y, \mathcal{T}) \to (X \times Y, \mathcal{T})$ is continuous. Thus by the universal property it follows that both $\pi_1 \circ \text{id} = \pi_1\colon (X \times Y, \mathcal{T}) \to (X, \mathcal{T}_X)$ and $\pi_2 \circ \text{id} = \pi_2\colon (X \times Y, \mathcal{T}) \to (Y, \mathcal{T}_Y)$ are continuous.

$$\begin{array}{ccc} & & (X \times Y, \mathcal{T}) \\ & \overset{\text{id}}{\nearrow} & \downarrow \pi_1 \\ (X \times Y, \mathcal{T}) & \xrightarrow[\pi_1 \circ \text{id}]{} & (X, \mathcal{T}_X) \end{array} \qquad \begin{array}{ccc} & & (X \times Y, \mathcal{T}) \\ & \overset{\text{id}}{\nearrow} & \downarrow \pi_2 \\ (X \times Y, \mathcal{T}) & \xrightarrow[\pi_2 \circ \text{id}]{} & (Y, \mathcal{T}_Y) \end{array}$$

Thus for each $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, we have

$$\pi_1^{-1}(U) = U \times Y \in \mathcal{T} \quad \text{and} \quad \pi_2^{-1}(V) = X \times V \in \mathcal{T},$$

and so, $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}$. Hence, $\mathcal{T}_{X\times Y} \subseteq \mathcal{T}$. Thus $\mathcal{T}_{X\times Y} = \mathcal{T}$. □

We can extend our discussion of the product topology from $X \times Y$ to $X_1 \times X_2 \times \cdots \times X_n$ where each X_i is a topological space. If we are to extend to the product $\prod_{\lambda\in\Lambda} X_\lambda$, which we can think of as the set of maps $f\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda$ where $f(\lambda) \in X_\lambda$ for each $\lambda \in \Lambda$, of an indexed family $\{X_\lambda\}_{\lambda\in\Lambda}$ of topological spaces we may proceed in two ways. We may equip $\prod_{\lambda\in\Lambda} X_\lambda$ with the topology generated by the basis $\prod_{\lambda\in\Lambda} U_\lambda$ where U_λ is open in X_λ for each $\lambda \in \Lambda$. This is known as the *box topology*. We may also equip $\prod_{\lambda\in\Lambda} X_\lambda$ with the topology generated by the subbasis $\mathcal{S} = \bigcup_{\mu\in\Lambda}\{\pi_\mu^{-1}(U_\mu) \mid U_\mu \text{ is open in } X_\mu\}$. This is known as the *product topology*. For finite products $\prod_{i=1}^n X_i$ the two topologies are the same. Also, the box topology is, in general, finer than the product topology. Finally, several results regarding finite products may be extended to arbitrary products when using the product topology but not the box topology.

5.3 Quotient spaces

Let X be a topological space. In Section 5.1, we discussed how to define the coarsest possible topology on a subset A of X such that the inclusion map $i: A \to X$ is continuous. This is known as the subspace topology. If we let A be a set which is not necessarily a subset of X and we consider a surjective map $\pi: X \to A$, the *quotient topology* is the finest topology on A such that π is continuous.

The torus $T^2 = S^1 \times S^1$ (see Figure 5.4) can be constructed by taking a rectangle and "gluing" its edges together in an appropriate way as shown in Figure 5.5. Such a construction involves the concept of quotient topology.

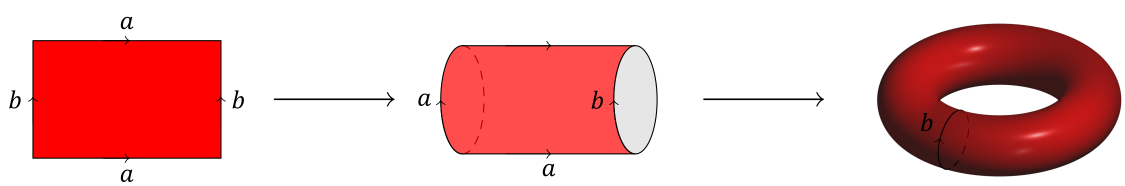


Figure 5.5: Constructing the torus T^2.

Definition 5.14 (Equivalence classes) Let X be a set, and let $\sim$ be an equivalence relation on X. The *equivalence class* of $x \in X$ is the subset

$$[x] = \{y \in X \mid x \sim y\}$$

of X. Let

$$X/\sim = \{[x] \mid x \in X\}$$

be the set of equivalence classes.

By definition, $x \in [x]$ for each $x \in X$ and $[x] = [y]$ if and only if $x \sim y$. Moreover, two equivalence classes $[x_1]$ and $[x_2]$ are either disjoint or equal. Finally, the union of all equivalence classes equal X.

Lemma 5.15 *Let X and A be sets, and let $\pi: X \to A$ be a surjective map. Then the map*

$$\varphi: X/\sim \to A$$

given by

$$\varphi([x]) = \pi(x),$$

where $x_1 \sim x_2$ if and only if $\pi(x_1) = \pi(x_2)$, is a bijection.

Proof. The map is well-defined since $[x_1] = [x_2]$ only if $x_1 \sim x_2$, and so, $\pi(x_1) = \pi(x_2)$ by definition of the equivalence relation. It is injective since $\varphi([x_1]) = \varphi([x_2])$ implies $\pi(x_1) = \pi(x_2)$, and so, $x_1 \sim x_2$, i.e., $[x_1] = [x_2]$. Finally, it is surjective since π is; every element of A is of the form $\pi(x) = \varphi([x])$ for some $x \in X$. □

Thus by Lemma 5.15 we can, up to a bijection, go back and forth between equivalence relations on X and surjective maps $X \to A$.

Definition 5.16 (Quotient space) Let X be a topological space, let A be a set, and let $\pi\colon X \to A$ be a surjective map. The *quotient topology on A induced by π* is the collection of subsets U of A such that $\pi^{-1}(U)$ is open in X. We say that π is a *quotient map* if A is given the quotient topology, and we call A the *quotient space*.

In other words, $\pi\colon X \to A$ is quotient map if it is surjective and a subset U of A is open in A if and only if $\pi^{-1}(U)$ is open in X. Equivalently, π is a quotient map if it is surjective and U is closed in A if and only if $\pi^{-1}(U)$ is closed in X. Clearly, a quotient map is continuous.

Lemma 5.17 *Let X be a topological space, let A be a set, and let $\pi\colon X \to A$ be a surjective map. Then the quotient topology on A induced by π is a topology and it is the finest topology on A such that π is continuous.*

Proof. Since $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(A) = X$, and both $\emptyset$ and X are open in X, it follows that $\emptyset$ and A are open in A. Thus T1 holds.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of subsets of A that are open in A. Then $\pi^{-1}(U_\lambda)$ is open in X for each $\lambda \in \Lambda$. Since

$$\pi^{-1}\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcup_{\lambda\in\Lambda} \pi^{-1}(U_\lambda)$$

is a union of open sets in X, it must be open in X. Hence, $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open in A. Thus T2 is satisfied.

Let U_1 and U_2 be subsets of A that are open in A. Then both $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are open in X. Since

$$\pi^{-1}(U_1 \cap U_2) = \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$$

is a (finite) intersection of open sets in X, it must be open in X. Hence, $U_1 \cap U_2$ is open in A, and so, T3 is satisfied. Thus the quotient topology is a topology.

Let $\mathcal{T}$ be a topology on A such that π is continuous. We must show that $\mathcal{T}$ is coarser than the quotient topology. Since π is continuous when A is given $\mathcal{T}$ as its topology, we have for each $V \in \mathcal{T}$ that $\pi^{-1}(V)$ is open in X, and so, V is in the quotient topology. Hence, $\mathcal{T}$ is coarser than the quotient topology. □

Example 5.18 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, let $A = \{a, b, c\}$, and let

$$\pi\colon \mathbb{R} \to A$$

be the map given by

$$\pi(x) = \begin{cases} a & x = 0, \\ b & x < 0, \\ c & x > 0. \end{cases}$$

Then the quotient topology on A induced by π is the collection $\{\emptyset, \{b\}, \{b, c\}, \{c\}, A\}$ of subsets of A.

Definition 5.19 (Open and closed maps) Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous map. We say that f is an *open map* if for each subset U of X that is open in X the image $f(U)$ is open in Y. Likewise, we say that f is a *closed map* if for each subset V of X that is closed in X the image $f(V)$ is closed in Y.

If $f\colon X \to Y$ is an open (closed) map and A is a subset of X then it is not true in general that $f|_A\colon A \to f(A)$ is an open (closed) map. However, if we also assume that A is an open (closed) subset of X, then $f|_A\colon A \to f(A)$ is an open (closed) map.

Consider the case where A is open in X and f is an open map. Let U be a subset of A that is open in A, i.e., $U = A \cap V$ for some open subset V of X. Then U is also open in X, and hence, $f|_A(U) = f(U)$ is an open subset of Y. Furthermore, since $f(A)$ is open in Y and $f(U) \subseteq f(A)$, $f(U)$ must be open in $f(A)$.

Example 5.20 Any homeomorphism is both open and closed. However, the converse is, in general, not true. Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and let $*$ be the space consisting of a single element. Then $\mathbb{R} \to *$ is both an open and a closed map but it is clearly not a homeomorphism.

Theorem 5.21 *Let X and Y be two topological spaces, and let $f\colon X \to Y$ be a bijective continuous map. Then the following are equivalent:*

(1) *f is a homeomorphism;*

(2) *f is an open map;*

(3) *f is a closed map.*

Proof. We will show that (1) is equivalent to (2) and that (2) is equivalent to (3).

Since f is a bijection, we have

$$\left(f^{-1}(U)\right)^{-1}(U) = f(U)$$

for any subset U of X. In particular, if U is an open subset of X and if we assume that f is a homeomorphism then $f(U)$ must be open in Y as f^{-1} is continuous. Hence, f is an open map. Furthermore, if we assume that f is open instead of being a homeomorphism then $f(U)$ is open in Y for U an open subset of X. Hence, $f^{-1}\colon Y \to X$ is continuous. Thus f is a homeomorphism. Hence, (1) is equivalent to (2).

Assume that f is an open map. Since f is a bijection, we have

$$f(X \setminus U) = f(U^c) = f(U)^c = Y \setminus f(U).$$

Thus f is closed: $X\setminus U$ is closed and $Y\setminus f(U)$ is closed. The opposite implication follows by a completely analogous argument. Hence, (2) is equivalent to (3).

This completes the proof. □

The following lemma establishes a relation between open and closed maps, and quotient maps.

Lemma 5.22 *Let X and Y be topological spaces, and let $\pi\colon X \to Y$ be a surjective continuous map.*

(1) *If π is in addition open then it is a quotient map.*

(2) *If π is in addition closed then it is a quotient map.*

Proof. Assume that π is in addition open. Let V be a subset of Y. If V is open in Y then $\pi^{-1}(V)$ is open in X by assumption of continuity of π. If $\pi^{-1}(V)$ is open in X then since π is surjective, we have

$$\pi(\pi^{-1}(V)) = V$$

which is open in Y since we have assumed that π is an open map. Hence, (1) holds.

Now assume that π is also closed in addition to being a surjective continuous map. Let W be a subset of Y. If W is closed in Y then $\pi^{-1}(W)$ is closed in X by assumption of continuity of π. If $\pi^{-1}(W)$ is closed in X then since π is surjective, we have

$$\pi(\pi^{-1}(W)) = W$$

which is closed in Y since we have assumed that π is a closed map. Hence, (2) holds. □

Example 5.23 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Consider $[0,1]$ as a subspace of $\mathbb{R}$ and S^1 as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is also given the standard topology. Let

$$\pi\colon [0,1] \to S^1$$

be the map given by

$$\pi(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Then, clearly, π is a surjective continuous map. We can show that π is also closed (to do this it helps to have defined compactness). Thus by Lemma 5.22 π is a quotient map. Note that π is not open as $\pi([0,t))$ is not open for $0 < t < 1$ (where we are using the fact that $[0,t)$ is open in $[0,1]$ for $0 < t < 1$).

Let $\sim$ be the equivalence relation on $[0,1]$ given by $s \sim t$ if and only if $\pi(s) = \pi(t)$, and let

$$p\colon [0,1] \to [0,1]/{\sim}$$

be the map given by

$$p(t) = [t].$$

Then, clearly, p is a surjective continuous map. The induced bijective map

$$\varphi\colon [0,1]/{\sim} \to S^1$$

given by

$$\varphi([t]) = \pi(t)$$

is then a homeomorphism from $[0,1]/{\sim}$ with the quotient topology induced by p to S^1 with the quotient topology induced by π. See Figure 5.6.

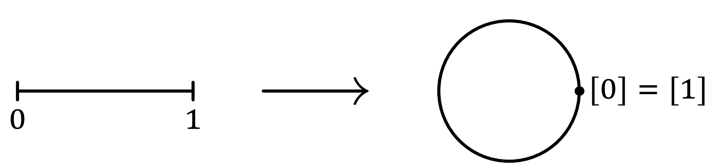


Figure 5.6: Constructing the circle S^1.

Example 5.24 Consider $[0,1] \times [0,1]$ as a subspace of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology, and $T^2 = S^1 \times S^1$ as a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ where $\mathbb{R}^4$ is also given the standard topology. Let

$$\pi\colon [0,1] \times [0,1] \to S^1 \times S^1$$

be the map given by

$$\pi(s,t) = (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t)).$$

Clearly, π is a surjective continuous function. As in the previous example, we can show that π is closed, and so, by Lemma 5.22 it is a quotient map.

Let $\sim$ be the equivalence relation on $[0,1] \times [0,1]$ given by $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$, and let

$$p\colon [0,1] \times [0,1] \to ([0,1] \times [0,1])/{\sim}$$

be the map given by

$$p(s,t) = [s,t].$$

Then, clearly, p is a surjective continuous map. The induced bijective map

$$\varphi\colon ([0,1] \times [0,1])/{\sim} \to S^1 \times S^1$$

given by

$$\varphi([s,t]) = \pi(s,t)$$

is then a homeomorphism from $([0,1] \times [0,1])/{\sim}$ with the quotient topology induced by p to $S^1 \times S^1$ with the quotient topology induced by π. The equivalence classes are then the sets

$$\{(0,0), (0,1), (1,0), (1,1)\}$$

(the four vertices of the square are identified),

$$\{(s,0), (s,1)\} \quad \text{and} \quad \{(0,t), (1,t)\}$$

(opposing edges of the square are identified), and

$$\{(s,t)\}$$

for $s, t \in (0,1)$. See Figure 5.5.

Example 5.25 The *real projective space* is the quotient space

$$\mathbb{R}\mathrm{P}^n = S^n/{\sim}$$

where S^n is the n-sphere and $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x = \pm y$, i.e., $[x] = \{x, -x\}$. We say that x and $-x$ are *antipodal* points. The topology on $\mathbb{R}\mathrm{P}^n$ is defined by the quotient map

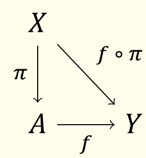
$$\pi\colon S^n \to \mathbb{R}\mathrm{P}^n$$

given by

$$\pi(x) = [x].$$

We end this section with an alternative description of the quotient topology.

Theorem 5.26 *Let X be a topological space, let A be a set, and let $\pi\colon X \to A$ be a surjective map. The quotient topology is the only topology on A with the following universal property: for every topological space Y and every map $f\colon A \to Y$, f is continuous if and only if $f \circ \pi\colon X \to Y$ is continuous.*



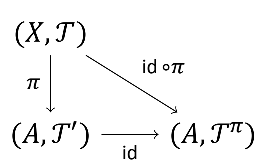
Proof. We follow the arguments for the proofs of Theorem 5.6 and Theorem 5.13 and adapt them to our current setting.

We first prove that the quotient topology $\mathcal{T}^\pi$ has the universal property that for every topological space Y and every map $f\colon A \to Y$, f is continuous if and only if $f \circ \pi\colon X \to Y$ is continuous.

Let A be given the quotient topology induced by π, and assume that $f\colon A \to Y$ is continuous. By definition of the quotient topology $\pi^{-1}(U)$ is open in X if and only if U is open in A. Hence, π is continuous. Thus by Theorem 3.16 $f \circ \pi\colon X \to Y$ is continuous. Now assume that $f \circ \pi\colon X \to Y$ is continuous. Let V be an open set in Y. Since π is a quotient map and $(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V))$ is open in X by assumption of continuity of $f \circ \pi$, it follows that $f^{-1}(V)$ is open in A. Thus f is continuous.

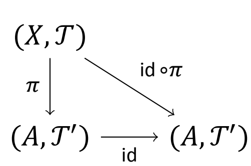
Let $\mathcal{T}'$ be a topology on A with the universal property that for every topological space Y and every map $f\colon A \to Y$, f is continuous if and only if $f \circ \pi\colon X \to Y$ is continuous. We must show that $\mathcal{T}' = \mathcal{T}^\pi$.

Let $\mathcal{T}$ be the topology on X, and let A be given the topology $\mathcal{T}'$. First let $Y = A$ with the quotient topology induced by $f \circ \pi$. Then for $f = \mathrm{id}\colon (A, \mathcal{T}') \to (A, \mathcal{T}^\pi)$, we have $\mathrm{id} \circ \pi = \pi\colon (X, \mathcal{T}) \to (A, \mathcal{T}^\pi)$ which is continuous. Hence, by the universal property id is continuous.



Thus any $V \in \mathcal{T}^\pi$ must also be an element of $\mathcal{T}'$, and so, $\mathcal{T}^\pi \subseteq \mathcal{T}'$.

Secondly let $Y = A$ with $\mathcal{T}'$ as its topology. Then, clearly, $f = \mathrm{id}\colon (A, \mathcal{T}') \to (A, \mathcal{T}')$ is continuous. Thus by the universal property it follows that $\mathrm{id} \circ \pi = \pi\colon (X, \mathcal{T}) \to (A, \mathcal{T}')$ is continuous.



Since the quotient topology induced by π is the finest topology on A such that π is a continuous map, cf. Lemma 5.17, it follows that $\mathcal{T}' \subseteq \mathcal{T}^\pi$. Hence, $\mathcal{T}' = \mathcal{T}^\pi$. □

5.4 Exercises

Exercise 5.1 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Show that any subspace of the form (a, b), i.e., an open interval, is homeomorphic to $\mathbb{R}$.

Exercise 5.2 Let X be a topological space and let Y be a subspace of X. If A is a subset of Y, show that the subspace topology on A inherited from Y is equal to the subspace topology on A inherited from X.

Exercise 5.3 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that the subset

$$A = \left\{x \in \mathbb{Q} \,\middle|\, -\sqrt{5} < x < \sqrt{5}\right\}$$

of $\mathbb{Q}$ is both open and closed in $\mathbb{Q}$.

Exercise 5.4 Let $X = \{a, b, c, d\}$ be given the topology $\mathcal{T}_X = \{\emptyset, \{a\}, \{a, c, d\}, \{c, d\}, X\}$, and let $Y = \{1, 2, 3\}$ be given the topology $\mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 3\}, Y\}$. Find a basis for the product topology on $X \times Y$.

Exercise 5.5 Let X and Y be topological spaces, and let A and B be subsets of X and Y, respectively. Show that the topology on $A \times B$ as a subspace of the product $X \times Y$ is equal to the product topology on $A \times B$ where A and B are given the subspace topology inherited from X and Y, respectively.

Exercise 5.6 Let X and Y be topological spaces. Show that the product topology is the coarsest topology on $X \times Y$ for which both of the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.

Exercise 5.7 Let X and Y be two topological spaces, and let $X \times Y$ be given the product topology. Show that if $f : X \to Y$ is a continuous map, the subspace

$$G = \{(x, y) \in X \times Y \mid y = f(x)\}$$

of $X \times Y$, is homeomorphic to X.

Exercise 5.8 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Let

$$\pi : \mathbb{R} \to \mathbb{Z}$$

be the map given by

$$\pi(x) = \begin{cases} x & x \in \mathbb{Z} \\ n & x \in (n-1, n+1), \text{and } n \text{ is an odd integer.} \end{cases}$$

Show that the quotient topology on $\mathbb{Z}$ induced by π is equal to the digital line topology, cf. Exercise 4.6.

6. Topological properties

6.1 Connected spaces

One of the fundamental results of calculus is the *intermediate value theorem*. The theorem says that for a continuous map $f: [a, b] \to \mathbb{R}$ and for a real number r between $f(a)$ and $f(b)$ there is a real number $c \in [a, b]$ such that $f(c) = r$. See Figure 6.1. From this result we can deduce that the graph of a continuous map (in this setting) is connected.

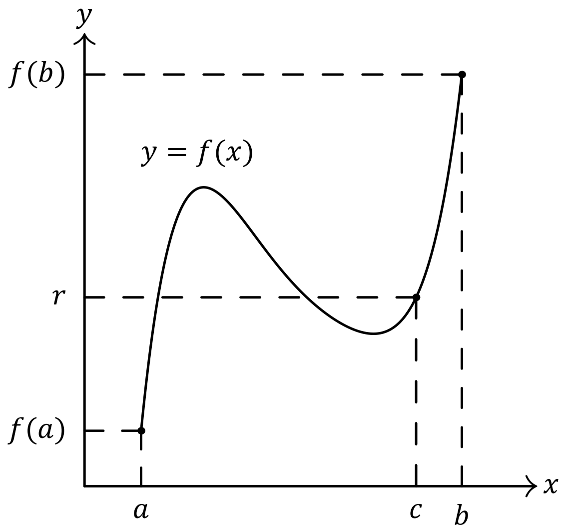


Figure 6.1: The intermediate value theorem.

Definition 6.1 (Connected space) Let X be a topological space. A *separation* of X is a pair of non-empty subsets U and V that are open in X, disjoint and whose union equal X. We say that X is *connected* if there are no separations of X. Otherwise it is *disconnected*.

The property of being connected is a topological property as it is formulated entirely in terms of open subsets. In other words, if X and Y are homeomorphic topological spaces and X is connected then so is Y.

Example 6.2 Let X be the set $\{a, b, c, d, e\}$. If we equip X with the topology $\mathcal{T}_1 = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d, e\}, \{d, e\}, X\}$ then it is disconnected; the pair $U = \{a, b, c\}$ and $V = \{d, e\}$ is a separation of X in this topology.

However, if we equip X with the topology $\mathcal{T}_2 = \{\emptyset, \{a, b, c\}, \{c\}, \{c, d, e\}, X\}$ then it is connected; there are no separations of X in this topology.

Example 6.3 Let X be an indiscrete space. Then X is connected as there are no separations of X, i.e., there are no non-empty open subsets of X who are disjoint and whose union equal X.

Example 6.4 Let X be a discrete space containing two or more points. Then X is disconnected. Let $p \in X$ and let $U = \{p\}$ and $V = U^c = X \setminus \{p\}$. Then the pair U and V is a separation of X.

Example 6.5 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and let $X = [0,1) \cup (1,2)$ be a subspace of $\mathbb{R}$. Since $U = [0,1)$ is open in X (but not in $\mathbb{R}$) and $V = (1,2)$ is open X, $U \cap V = \emptyset$ and $U \cup V = X$, they form a separation of X. Thus X is disconnected.

In the above examples the spaces that are connected all share the property that the only subsets that are both open and closed in X are $\emptyset$ and X. Likewise, the disconnected spaces all share the property that there are non-empty proper subsets of X that are both open and closed in X.

Theorem 6.6 *Let X be a topological space. Then X is connected if and only if the are no non-empty proper subsets of X that are both open and closed in X.*

Proof. We prove the equivalent statement that X is disconnected if and only if there are non-empty proper subsets of X that are both open and closed in X.

Assume X is disconnected, i.e., that there is a separation of X. Let U and V be a separation of X. Thus U is open in X. Since $U \cap V = \emptyset$ and $U \cup V = X$, we have $U^c = X \setminus U = V$. Thus U^c is open in X, and so, U is closed in X. Hence, U is both open and closed in X. Likewise, V is both open and closed in X.

Assume that the non-empty proper subset U of X is both open and closed in X. Let $V = U^c = X \setminus U$. Then V is open in X, $U \cap V = \emptyset$ and $U \cup V = X$. Hence, the pair U and V is a separation of X. Thus X is disconnected. □

Theorem 6.7 *Let X be a connected space, Y be a topological space, and let $f : X \to Y$ be a surjective continuous map. Then Y is connected.*

Proof. We prove the equivalent statement that if $f : X \to Y$ is a surjective continuous map and Y is disconnected then X is disconnected.

Assume that Y is disconnected, i.e., there is a separation of Y. Let the pair U and V be a separation of Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty subsets of X which are open in X as f is a surjective continuous map. Furthermore,

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset \qquad \text{and} \qquad f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = X.$$

Hence, the pair $f^{-1}(U)$ and $f^{-1}(V)$ is a separation of X. Thus X is disconnected. □

A subset A of a topological space Z is *connected* if A is connected in the subspace topology. Thus the theorem can be extended to saying that the continuous image of a connected space is connected, i.e., assuming X is a connected space and $f : X \to Y$ is a continuous map then $f(X)$ is connected in Y.

We may also describe connectedness by way of the following theorem.

Theorem 6.8 *Let X be a topological space. Then X is connected if and only if every continuous map form X to a discrete space, with at least two points, is constant.*

Proof. We prove the equivalent statement that X is disconnected if and only if there is a non-constant continuous map from X to a discrete space (with at least two points).

Assume that X is disconnected. Let U be a non-empty proper subset of X that is both open and closed in X. Let $Y = \{a, b\}$ be given the discrete topology. The map $f: X \to Y$ that sends U to a and $U^c = X \setminus U$ to b is continuous and not constant.

Let Y be a discrete space with at least two points. Assume that $f: X \to Y$ is a non-constant continuous map. Then for each $y \in Y$, $f^{-1}(\{y\})$ is a non-empty proper subset of X that is both open and closed in X. Thus by Theorem 6.6, X is disconnected. □

We will prove that the (finite) product of connected spaces is again connected. To prove this we will need the following two results.

Lemma 6.9 *Let X be a disconnected space with separation U and V, and let A be a connected subspace of X. Then $A \subseteq U$ or $A \subseteq V$.*

Proof. Since U and V are open in X, the intersections $A \cap U$ and $A \cap V$ are both open in A (in the subspace topology). Furthermore, the complement of $A \cap U$ in A is equal to $A \cap V$ as $U^c = X \setminus U = V$. Hence, $A \cap U$ is also closed in A. Thus by Theorem 6.6, $A \cap U$ is either empty or all of A as A is connected. If $A \cap U = \emptyset$ then $A \subseteq V$. If $A \cap U = A$ then $A \subseteq U$. □

Theorem 6.10 *Let X be a topological space, and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of connected subspaces of X such that $\bigcap_{\lambda \in \Lambda} A_\lambda$ is non-empty. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.*

Proof. Let $Y = \bigcup_{\lambda \in \Lambda} A_\lambda$. Suppose that Y is disconnected, i.e., that there is a separation of Y. Let U and V be a separation of Y. We will show that this leads to a contradiction, and so, Y must be connected.

Let $p \in \bigcap_{\lambda \in \Lambda} A_\lambda$. Then either $p \in U$ or $p \in V$. Assume without loss of generality that $p \in U$. By Lemma 6.9 it follows that for each $\lambda \in \Lambda$ either $A_\lambda \subseteq U$ or $A_\lambda \subseteq V$. Since we have assumed that $p \in U$ we must have $A_\lambda \subseteq U$ for all $\lambda \in \Lambda$. Thus $Y \subseteq U$. But this implies that V is empty, and hence, contradicts that U and V is a separation of Y. Thus Y is connected. □

We can prove that the (finite) product of connected spaces is again connected.

Theorem 6.11 *Let $X_1, X_2, \dots, X_n$ be connected spaces. Then the product space $X_1 \times X_2 \times \cdots \times X_n$ is connected.*

Proof. We prove the statement for the product of two connected spaces. The general result then follows by an induction argument.

Let X and Y be two connected spaces. We must prove that $X \times Y$ is connected. Since for each $x \in X$ the subspace $\{x\} \times Y$ of $X \times Y$ is homeomorphic to Y, it follows that $\{x\} \times Y$ is connected. Similarly, for each $y \in Y$ the subspace $X \times \{y\}$ is homeomorphic to X, and hence, $X \times \{y\}$ is connected. Thus by Theorem 6.10 it follows that for each $x \in X$ and each $y \in Y$ the subspace $(\{x\} \times Y) \cup (X \times \{y\})$

is connected as it is the union of two connected spaces whose intersection is $(\{x\} \times Y) \cap (X \times \{y\}) = \{(x, y)\} \neq \emptyset$.

Fix $x_0 \in X$ and let $A_y = (\{x_0\} \times Y) \cup (X \times \{y\})$. Then for each $y \in Y$ the subspace A_y of $X \times Y$ is connected as it is the union of two connected spaces whose intersection is equal to $\{(x_0, y)\}$. Hence, by Theorem 6.10 it follows that $\bigcup_{y \in Y} A_y$ is connected as it is the union of connected spaces whose intersection in non-empty. Since

$$\bigcup_{y \in Y} A_y = X \times Y,$$

it follows that $X \times Y$ is connected. □

The theorem can be extended to hold for arbitrary products, $\prod_{\lambda \in \Lambda} X_\lambda$, if we equip the product with the product topology. If we equip $\prod_{\lambda \in \Lambda} X_\lambda$ with the box topology the statement is no longer true.

An important example of a connected space is the set of real numbers equipped with the standard topology.

Theorem 6.12 (The real numbers are connected) *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then $\mathbb{R}$ is connected.*

We will use the fact that the real numbers satisfy the following two properties.

(1) Every subset of $\mathbb{R}$ that is bounded above has a least upper bound. This is known as the *least upper bound property*.

(2) If $x, y \in \mathbb{R}$ with $x < y$ then there is a real number z such that $x < z < y$.

Proof. Assume that $\mathbb{R}$ is disconnected, i.e., that there is a separation of $\mathbb{R}$. Let U and V be a separation of $\mathbb{R}$, and choose $a \in U$ and $b \in V$. We may assume without loss of generality that $a < b$.

Let $A = [a, b] \cap U$ and $B = [a, b] \cap V$. Then the pair A and B is a separation of $[a, b]$, with $a \in A$ and $b \in B$. Also note that A is bounded above by b. Hence, by the least upper bound property A has a least upper bound; $c = \sup A$. Thus $a \leqslant c \leqslant b$. We will show that c belongs neither to A nor to B thus contradicting the fact that A and B is a separation of $[a, b]$.

Assume that $c \in B$. Since $a \notin B$ and B is open in $[a, b]$, it follows that there is a real number d such that $a < d < c$ and $(d, c] \subseteq B$. This implies that d is an upper bound of A and that d is less than the least upper bound c. That is a contradiction, and so, $c \notin B$.

Now assume that $c \in A$. Since A is open in $[a, b]$ and $b \notin A$, there is a real number d such that $[c, d) \subseteq A$. For any $e \in (c, d)$ it follows that $e \in A$ and $e > c$. That is a contradiction to the fact that c is an upper bound of A. Thus $c \notin A$.

Hence, $c \notin A$ and $c \notin B$. This is a contradiction to the fact that $c \in [a, b]$ and that A and B is a separation of $[a, b]$. Thus $\mathbb{R}$ must be connected. □

As an immediate consequence of Theorem 6.12, we get that open intervals of the form (a, b), $(-\infty, b)$ and (a, ∞) are all connected as they are all homeomorphic to $\mathbb{R}$ (with the standard topology). We can also show that every closed interval $[a, b]$ is connected. Furthermore, by Theorem 6.11 and Theorem 6.12 $\mathbb{R}^n$ is a connected space.

Theorem 6.13 **(Generalized intermediate value theorem)** *Let X be a connected space, and let $f\colon X \to \mathbb{R}$ be a continuous map where $\mathbb{R}$ is given the standard topology. If $a, b \in X$ and if r is a real number that lies between $f(a)$ and $f(b)$, there is a $c \in X$ such that $f(c) = r$.*

Proof. Assume that $r \notin f(X)$. We will show that this contradicts the assumption that X is connected.

By the assumption that $r \notin f(X)$, i.e., $f(X) \subseteq \mathbb{R} \setminus \{r\} = (-\infty, r) \cup (r, \infty)$, we have a separation of X:

$$U = f^{-1}((-\infty, r)) \qquad \text{and} \qquad V = f^{-1}((r, \infty))$$

are disjoint non-empty open subsets of X whose union equals X. Thus they are a separation of X. This contradicts the assumption that X is connected. Hence, $r \in f(X)$. In other words, there is a $c \in X$ such that $f(c) = r$. □

We end this section with a discussion of path connectivity.

Definition 6.14 **(Path connected space)** Let X be a topological space, and let $x, y \in X$. A *path* from x to y is a continuous map $f\colon [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$ where $[a, b]$ is a subspace of $\mathbb{R}$ with the standard topology. We say that X is *path connected* if every pair of points of X can be joined by a path in X.

The property of being path connected is a topological property as it is completely described using the elements of X and its open subsets. In other words, if X and Y are homeomorphic topological spaces and X is path connected then so is Y.

Example 6.15 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then $\mathbb{R}$ is path connected as for any two points, $p, q \in \mathbb{R}$, there is a path from p to q in $\mathbb{R}$, e.g., the path given by $f(t) = (1 - t)p + tq$ where $t \in [0, 1]$.

Example 6.16 For all $n \geqslant 2$, $\mathbb{R}^n$ with the standard topology is path connected, and so is $\mathbb{R}^n \setminus \{p\}$ for each $p \in \mathbb{R}^n$. For $n = 1$, $\mathbb{R} \setminus \{p\}$ is not (path) connected.

The next theorem states that path connectedness implies connectedness. While the converse is not true in general, it is true that open subsets of $\mathbb{R}^n$ that are connected (where $\mathbb{R}^n$ is given the standard topology) are also path connected. See [6, Proposition 12.25] for a proof of this fact.

Theorem 6.17 **(Path connectedness implies connectedness)** *Let X be a path connected space. Then X is connected.*

Proof. Assume that X is path connected but that X is disconnected, i.e., there is a separation of X. Let U and V be a separation of X. Let $f\colon [a, b] \to X$ be a path from $x \in U$ to $y \in V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint non-empty open subsets of $[a, b]$ whose union is equal to $[a, b]$. Thus they are a separation of $[a, b]$. This is a contradiction to the fact that $[a, b]$ is a connected space. □

Example 6.18 (The topologist's sine curve) The *topologist's sine curve* is the subspace

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, 0 < x \leqslant 1 \right\} \cup \{(0, y) \mid -1 \leqslant y \leqslant 1\}$$

of $\mathbb{R}^2$ with the standard topology. It can be shown that S is connected but not path connected. For a proof of this fact, see [4, pp. 156–157]. See Figure 6.2.

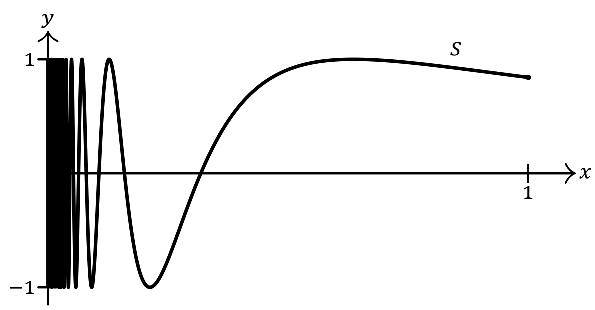


Figure 6.2: The topologist's sine curve.

6.2 Hausdorff spaces

A common feature that we typically want a topological space to have is the ability to separate the individual points. This is commonly referred to as *separation axioms*. We will in this section focus on the most common separation axiom.

Definition 6.19 Let X be a topological space. We say that X is *Hausdorff* if for each pair of points $x, y \in X$, with $x \neq y$, there are disjoint neighborhoods U and V of x and y, respectively. In other words, for each pair of distinct points $x, y \in X$ there are open subsets U and V of X with $x \in U$ and $y \in V$ where $U \cap V = \emptyset$.

The property to be a Hausdorff space is completely described using the elements of X and its open subsets, and so, it is a topological property. In other words, if X and Y are homeomorphic topological spaces and X is Hausdorff then so is Y. To see this let X be a Hausdorff space and let y_1 and y_2 be distinct points in Y. If $f: X \to Y$ is a homeomorphism then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are distinct points in X. Since X is Hausdorff this means that there are disjoint neighborhoods U_1 and U_2 of $f^{-1}(y_1)$ and $f^{-1}(y_2)$, respectively. Thus $f(U_1)$ and $f(U_2)$ are disjoint neighborhoods of y_1 and y_2, respectively. Hence, Y is Hausdorff.

Example 6.20 Let X be the set $\{a, b, c\}$. If we equip X with the discrete topology then X is Hausdorff since for all pairs of distinct points $x, y \in X$, the open subsets $\{x\}$ and $\{y\}$ are neighborhoods of x and y, respectively, and $\{x\} \cap \{y\} = \emptyset$.

However, if we equip X with the topology $\{\emptyset, \{a, b\}, \{b\}, X\}$ it is *not* Hausdorff; the only neighborhood of c is $U = X$, and no neighborhood V of either a or b can be disjoint from X.

Example 6.21 Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and X equipped with $\mathcal{T}_1$ as its topology is Hausdorff then X equipped with $\mathcal{T}_2$ as its topology is also Hausdorff.

Theorem 6.22 *Every metric space is Hausdorff.*

Proof. Let (X, d) be a metric space, and let x and y be two distinct points in X. We must show that there are disjoint neighborhoods U and V of x and y, respectively.

Let $\delta = d(x, y)$. Then $\delta > 0$. Let $U = \mathrm{B}(x; \delta/2)$ and $V = \mathrm{B}(y; \delta/2)$ be neighborhoods of x and y, respectively. Then by M3, $U \cap V = \emptyset$. □

Example 6.23 For all integers $n \geqslant 1$, the metric space $(\mathbb{R}^n, d)$ is Hausdorff. In particular, $\mathbb{R}^n$ with the standard topology is Hausdorff.

All Hausdorff spaces share the property that finite subsets are closed which is an immediate consequence of the following theorem.

Theorem 6.24 *Let X be a Hausdorff space. Then for each $x \in X$ the subset $\{x\}$ of X is closed in X.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is Hausdorff, we have neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Then $x \notin V$. In other words, $x \in V^c = X \setminus V$. Since V is open, V^c is closed in X. Thus $\overline{\{x\}} \subseteq V^c$, and hence, $y \notin \overline{\{x\}}$. Hence, $\{x\} = \overline{\{x\}}$. Thus $\{x\}$ is closed in X. □

There are examples of topological spaces who are not Hausdorff but have the property that finite subsets are closed. One such example is the set of real numbers equipped with the cofinite topology.

We have seen that the (finite) product of connected spaces is connected, cf. Theorem 6.11. The same statement holds for Hausdorff spaces, i.e., the (finite) product of Hausdorff spaces is Hausdorff.

Theorem 6.25 *Let $X_1, X_2, \dots, X_n$ be Hausdorff spaces. Then the product space $X_1 \times X_2 \times \cdots \times X_n$ is Hausdorff.*

Proof. We prove the statement for two Hausdorff spaces. The general result then follows from an induction argument.

Let (x_1, y_1) and (x_2, y_2) be two distinct points in $X \times Y$, i.e., $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ there must be neighborhoods U_X and V_X in X of x_1 and x_2, respectively, such that $U_X \cap V_X = \emptyset$ as X is assumed to be Hausdorff. Then $U_X \times Y$ and $V_X \times Y$ are neighborhoods of (x_1, y_1) and (x_2, y_2), respectively, where $(U_X \times Y) \cap (V_X \times Y) = \emptyset$. Similarly, if $y_1 \neq y_2$ there must be neighborhoods U_Y and V_Y in Y of y_1 and y_2, respectively, such that $U_Y \cap V_Y = \emptyset$ as Y is assumed to be Hausdorff. Then $X \times U_Y$ and $X \times V_Y$ are neighborhoods of (x_1, y_1) and (x_2, y_2), respectively, where $(X \times U_Y) \cap (X \times V_Y) = \emptyset$. Hence, $X \times Y$ is Hausdorff. □

The theorem be extended to hold for arbitrary products, $\prod_{\lambda \in \Lambda} X_\lambda$, if we equip the product with either the product topology or the box topology.

We end this section with a result that helps us decide whether or not a topological space is Hausdorff.

Theorem 6.26 *Let X be a topological space. Then X is Hausdorff if and only if the* diagonal *$\Delta = \{(x, x) \mid x \in X\}$ is closed in the product space $X \times X$.*

Proof. Assume that X is a Hausdorff space. Thus for any two distinct points x and y in X there are neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Thus $U \times V$ is open in $X \times X$ and $(x, y) \in U \times V$, and so, $(U \times V) \cap \Delta = \emptyset$. Hence, there is a neighborhood $N_{(x,y)}$ of (x, y) such that $N_{(x,y)} \subseteq \Delta^c = (X \times X) \setminus \Delta$. Thus by Theorem 3.10, it follows that Δ^c is open in $X \times X$, and so, Δ is closed in $X \times X$.

Now assume that Δ is closed in the product space $X \times X$. Then for any point $(x, y) \in X \times X$ with $x \neq y$, i.e., $(x, y) \in \Delta^c$, there is a basis element $U \times V$ for the product topology on $X \times X$ such that $(x, y) \in U \times V \subseteq \Delta^c$. Since $U \times V \subseteq \Delta^c$, we have $U \cap V = \emptyset$. Thus U and V are neighborhoods of x and y, respectively, such that $U \cap V = \emptyset$. Hence, X is Hausdorff. □

6.3 Compact spaces

In Section 6.1 we saw how the intermediate value theorem might be generalized to connected spaces. In this section we will see how the *extreme value theorem* may be generalized to compact spaces. The extreme value theorem says that for a continuous map $f\colon [a, b] \to \mathbb{R}$ there are points $m, M \in [a, b]$ such that $f(m) \leqslant f(x) \leqslant f(M)$ for all $x \in [a, b]$. See Figure 6.3.

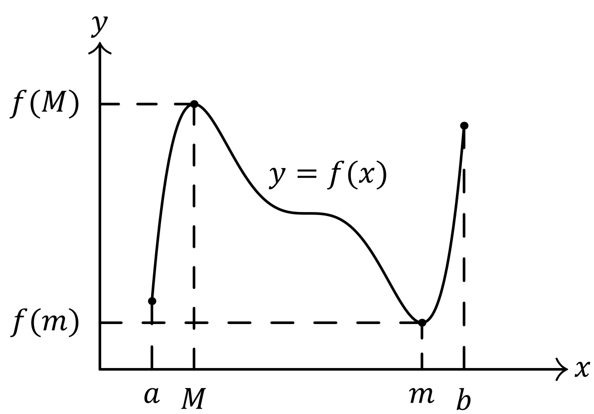


Figure 6.3: The extreme value theorem.

Definition 6.27 (Cover of a space) Let X be a topological space, and let $\mathcal{A}$ be a collection of subsets of X. We say that $\mathcal{A}$ is a *cover* of X, or *covering* of X if $X = \bigcup_{A \in \mathcal{A}} A$. If A is also open in X for each $A \in \mathcal{A}$, we say that $\mathcal{A}$ is an *open cover* of X, or *open covering* of X. We say that $\mathcal{A}'$ is a *subcover* of $\mathcal{A}$ if $\mathcal{A}'$ is another cover of X that satisfies $\mathcal{A}' \subseteq \mathcal{A}$.

Example 6.28 Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}$ is an open cover of X. Similarly, if $\mathcal{S}$ is a subbasis for the topology on X, then $\mathcal{S}$ is an open cover of X.

Definition 6.29 **(Compact spaces)** Let X be a topological space. We say that X is *compact* if every open cover $\mathcal{A}$ of X contains a finite subcover.

The property of being compact is a topological property as it is formulated entirely in terms of the collection of open sets. In other words, if X and Y are homeomorphic topological spaces and X is compact then so is Y.

Example 6.30 Let X be a finite topological space. Then X is compact as there are only finitely many different open subsets A of X, and so, any collection covering X must necessarily be finite.

Example 6.31 Let X be an indiscrete space. Then X is compact as the only open covers are the collections $\{X\}$ and $\{\emptyset, X\}$ which are finite.

Example 6.32 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Since the open cover

$$\mathcal{A} = \{(n-1, n+1) \mid n \in \mathbb{Z}\},$$

does not admit a finite subcover, $\mathbb{R}$ is *not* compact.

Definition 6.33 **(Compact subspaces)** Let X be a topological space, and let A be a subset of X. We say that A is *compact* if A is compact in the subspace topology.

If A is a subspace of X, a collection $\mathcal{A}$ of subsets of X is a *cover* of A if the union of elements of $\mathcal{A}$ *contains* A.

Lemma 6.34 *Let X be a topological space, and let A be a subspace of X. Then A is compact if and only if every cover of A by open subsets of X contains a finite subcollection that covers A.*

Proof. Assume that A is compact. Let $\mathcal{C}$ be a cover of A by open subsets of X. Then the collection

$$\mathcal{C}' = \{A \cap U \mid U \in \mathcal{C}\}$$

is an open cover of A. Since A is compact there must be a finite subcover $\{A \cap U_1, A \cap U_2, \ldots, A \cap U_n\}$ of $\mathcal{C}'$. Hence, $\{U_1, U_2, \ldots, U_n\}$ is a finite subcollection of $\mathcal{C}$ that covers A.

Now assume that every cover of A by open subsets of X contains a finite subcollection that covers A. Let $\mathcal{C} = \{V_\lambda\}_{\lambda \in \Lambda}$ be a cover of A by open subsets of A. Hence, by definition of the subspace topology, cf. Definition 5.1, we have for each $\lambda \in \Lambda$ that $V_\lambda = A \cap U_\lambda$ where U_λ is an open subset of X. Thus the collection $\mathcal{C}' = \{U_\lambda\}_{\lambda \in \Lambda}$ is a cover of A by open subsets of X. Then, by assumption, there must be a finite subcollection $\{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_n}\}$ that covers A. Hence, $\{V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n}\}$ is a finite subcover of $\mathcal{C}$. Thus every cover of A by open subsets of A has a finite subcover, and so, A is compact. □

The following two theorems indicates that being compact and being closed are closely related properties.

Theorem 6.35 *Let X be a compact space, and let A be a closed subset of X. Then A is compact.*

Proof. Let $\mathcal{C}$ be a cover of A, i.e., $A \subseteq \bigcup_{C\in\mathcal{C}} C$, by open subsets of X. Since A is closed in X, $A^c = X \setminus A$ is open in X. Thus

$$\mathcal{A} = \mathcal{C} \cup \{A^c\}$$

is an open cover of X. Since X is compact there must be a finite subcover $\mathcal{A}' \subseteq \mathcal{A}$ of X. If $\mathcal{A}'$ contains A^c, let $\mathcal{A}'' = \mathcal{A}' \setminus \{A^c\}$. Then $\mathcal{A}''$ is a finite subcover of $\mathcal{C}$ that covers A. If $\mathcal{A}'$ does not contain A^c then $\mathcal{A}'$ is a finite subcover of $\mathcal{C}$ that covers A. Either way there is a finite subcover of $\mathcal{C}$ that covers A. Thus A is compact. □

Theorem 6.36 *Let X be a Hausdorff space, and let K be a subset of X which is compact. Then K is closed in X.*

Proof. We show that $K^c = X \setminus K$ is open in X. Let $x \in K^c$. Then for each $y \in K$ there are neighborhoods U_y and V_y of x and y, respectively, such that $U_y \cap V_y = \emptyset$, since X is assumed to be Hausdorff and $x \neq y$.

The collection $\mathcal{C} = \{V_y \mid y \in K\}$ of open subsets of X covers K since

$$K \subseteq \bigcup_{y\in K} V_y.$$

Since K is assumed to be compact there must be a finite subcollection $\{V_{y_1}, V_{y_2}, \dots, V_{y_n}\}$ that covers K. Let $V = V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$, and let $U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$. Then U is open in X, $x \in U$ and $U \cap V = \emptyset$. Furthermore, $U \subseteq X \setminus V \subseteq X \setminus K = K^c$. Hence, K^c is open in X. □

Theorem 6.37 *Let X be a compact space, Y a topological space and let $f\colon X \to Y$ be a surjective continuous map. Then Y is compact.*

Proof. Let $\mathcal{C} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of Y. Then $\mathcal{A} = \{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of X. Since X is compact there must be a finite subcover $\mathcal{A}' = \{f^{-1}(U_{\lambda_1}), f^{-1}(U_{\lambda_2}), \dots, f^{-1}(U_{\lambda_n})\}$ of $\mathcal{A}$. Then $\mathcal{C}' = \{U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_n}\}$ is a finite subcover of $\mathcal{C}$. Hence, Y is compact. □

The theorem can be extended to saying that the continuous image of a compact space is compact, i.e., assuming X is a compact space and $f\colon X \to Y$ is a continuous map then $f(X)$ is compact.

We will prove that the (finite) product of compact spaces is compact. To prove this we need the following result.

Lemma 6.38 **(Tube lemma)** *Let X be a topological space, and let Y be a compact space. If $x \in X$ and U is an open set in the product space $X \times Y$ containing $\{x\} \times Y$, then there is a neighborhood W of x in X such that $W \times Y \subseteq U$.*

The set $W \times Y$ is often called a *tube* about $\{x\} \times Y$. See Figure 6.4.

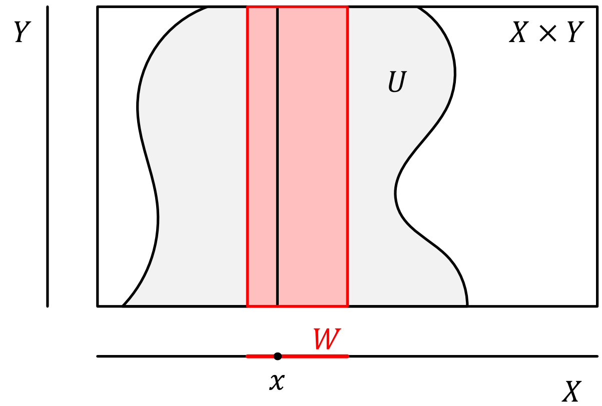


Figure 6.4: A tube about $\{x\} \times Y$.

Proof. As U is open in $X \times Y$ and $(x, y) \in \{x\} \times Y \subseteq U$ for all $y \in Y$, there is a basis element $W_y \times V_y \subseteq U$ for the product topology on $X \times Y$ such that $(x, y) \in W_y \times V_y$. The collection $\{V_y\}_{y \in Y}$ is an open cover of Y. Since Y is compact there must be a finite subcover of $\{V_y\}_{y \in Y}$, say, $\{V_{y_1}, V_{y_2}, \dots, V_{y_n}\}$.

Let

$$W = \bigcap_{i=1}^{n} W_{y_i}.$$

Then W is open in X, and it must contain x. Clearly,

$$W \times Y \subseteq \bigcup_{i=1}^{n} (W_{y_i} \times V_{y_i}) \subseteq U.$$

Thus $\{x\} \times Y \subseteq W \times Y$ and $W \times Y \subseteq U$. □

The lemma is not true if we remove the assumption that Y is compact; the open set

$$U = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |xy| < 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

does contain $\{0\} \times \mathbb{R}$ but does not contain any tube $W \times \mathbb{R}$ containing $\{0\} \times \mathbb{R}$. (Here we have assumed that $\mathbb{R}$ is given the standard topology, and hence, it is not compact, cf. Example 6.32.)

Theorem 6.39 *Let $X_1, X_2, \dots, X_n$ be compact spaces. Then the product space $X_1 \times X_2 \times \cdots \times X_n$ is compact.*

Proof. We prove the statement for the product of two compact spaces. The general result then follows by an inductive argument.

Let X and Y be compact spaces. Let $\mathcal{A}$ be an open cover of $X \times Y$. We must show that there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$. For each $x \in X$, $\{x\} \times Y$ is compact as it is homeomorphic to Y which is assumed to be compact. Thus there is a finite subcollection $\mathcal{A}_x$ of $\mathcal{A}$ that covers $\{x\} \times Y$. Let $U_x = \bigcup_{A_x \in \mathcal{A}_x} A_x$. Then U_x is open in $X \times Y$ and contains $\{x\} \times Y$. Thus by Lemma 6.38 for each $x \in X$ there is a neighborhood $W_x \subseteq X$ such that $x \in W_x$ and $W_x \times Y \subseteq U_x$. Furthermore, $\mathcal{A}_x$ covers $W_x \times Y$.

Now let $x \in X$ vary. The collection $\{W_x\}_{x \in X}$ is then an open cover of X. Since X is compact, there must be a finite subcover $\{W_{x_1}, W_{x_2}, \dots, W_{x_n}\}$ of $\{W_x\}_{x \in X}$. For each $1 \leqslant i \leqslant n$ the subspace $W_{x_i} \times Y$

is covered by the finite subcollection $\mathcal{A}_{x_i}$ of $\mathcal{A}$. Hence,

$$X \times Y = \bigcup_{i=1}^{n} W_{x_i} \times Y$$

is covered by the subcollection $\mathcal{A}' = \bigcup_{i=1}^{n} \mathcal{A}_{x_i}$ of $\mathcal{A}$. Thus $X \times Y$ is compact. □

The theorem can be extended to hold for arbitrary products of compact spaces if we equip the product with the product topology. This is known as *Tychonoff's theorem*. It is a deep result whose proof uses several original ideas. If we equip the product with the box topology the statement is no longer true.

We have already seen that the real line (with the standard topology) is not compact, cf. Example 6.32. The next theorem states that all closed intervals of the real line are compact.

Theorem 6.40 *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then every closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Proof. Let $\mathcal{A} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of $[a, b]$ by open subsets of $\mathbb{R}$, and let

$$S = \{x \in [a, b] \mid [a, x] \text{ is covered by a finite subcollection of } \mathcal{A}\}.$$

Note that S is bounded above by b. Since $a \in U_\lambda$ for some $\lambda \in \Lambda$, the singleton $\{U_\lambda\}$ is a finite subcollection of $\mathcal{A}$ that covers $[a, a] = \{a\}$. Hence, S is non-empty and it is bounded above. Thus S has a least upper bound; $c = \sup S$. Clearly, $a \leqslant c \leqslant b$.

We will show that $c \in S$. The result follows if we can also show that $c = b$. Choose $\lambda' \in \Lambda$ with $c \in U_{\lambda'}$. Since $U_{\lambda'}$ is open in $\mathbb{R}$ there is a real number $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \subseteq U_{\lambda'}$. Hence, there is an $x \in S$ such that $c - \epsilon < x$. So by definition of S there is a finite subcollection $\{U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_n}\}$ of $\mathcal{A}$ such that $[a, x] \subseteq \bigcup_{i=1}^{n} U_{\lambda_i}$. Furthermore, $[x, c] \subseteq U_{\lambda'}$. Thus $[a, c] = [a, x] \cup [x, c]$ is covered by the finite subcollection $\{U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_n}, U_{\lambda'}\}$ of $\mathcal{A}$. Hence, $c \in S$.

We now show that $c = b$. Assume that $c < b$. Then there must be a $y \in [a, b]$ such that $c < y < c + \epsilon$. Thus $[a, y]$ is covered by the subcollection $\{U_{\lambda_1}, U_{\lambda_2}, \dots, U_{\lambda_n}, U_{\lambda'}\}$ of $\mathcal{A}$ such that $y \in S$. This is a contradiction of the fact that c is an upper bound. Hence, $c = b$. □

In order to state and prove the Heine–Borel theorem we need the following definition.

Definition 6.41 **(Bounded subsets)** Let (X, d) be a metric space, and let A be a subset of X. We say that A is *bounded* if there is an $M \in \mathbb{R}$ such that $d(a_1, a_2) \leqslant M$ for all $a_1, a_2 \in A$.

Equivalently, we may say that a subset A of a metric space (X, d) is *bounded* if there is a $K \in \mathbb{R}$ and $x \in X$ such that $d(a, x) \leqslant K$ for all $a \in A$. In particular, this means that a subset of $\mathbb{R}^n$ equipped with the Euclidean metric is bounded if it is contained in some open ball of finite radius centered at the origin.

Theorem 6.42 **(Heine–Borel)** *Let $\mathbb{R}^n$ be given the (Euclidean) metric topology and the Euclidean metric. A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Assume that A is compact. By Theorem 6.22, $\mathbb{R}^n$ is Hausdorff, and so, by Theorem 6.36 A is closed in $\mathbb{R}^n$. We must show that A is also bounded. Let $\mathcal{A} = \{\mathrm{B}(0; n) \mid n \in \mathbb{Z}_+\}$, i.e., a collection of open balls centered at the origin in $\mathbb{R}^n$. Then $\mathcal{A}$ is a cover of A by open subsets of $\mathbb{R}^n$. Since A is compact there must be a finite subcollection of $\mathcal{A}$ that covers A. Thus there is an $N \in \mathbb{Z}_+$ such that $A \subseteq \mathrm{B}(0; N)$. Hence, for all $a_1, a_2 \in A$ we have $d(a_1, a_2) < 2N$. Thus A is bounded.

Now assume that A is bounded and closed in $\mathbb{R}^n$. Let $p = (p_1, p_2, \dots, p_n) \in A$. Since A is bounded there is an $M \in \mathbb{R}$ such that $d(a_1, a_2) \leqslant M$ for all $a_1, a_2 \in A$. Then A is contained in the product space

$$P = [p_1 - M, p_1 + M] \times [p_2 - M, p_2 + M] \times \cdots \times [p_n - M, p_n + M]$$

which by Theorem 6.40 and Theorem 6.39 is compact. Since A is closed in $\mathbb{R}^n$ and a subset of P, it follows by extending Theorem 6.35 to our setting that A is compact. Specifically, let $\mathcal{C}$ be a cover of A by open subsets of $\mathbb{R}^n$. Since A is closed in $\mathbb{R}^n$, $A^c = \mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$. Thus

$$\mathcal{A} = \mathcal{C} \cup \{A^c\}$$

is an open cover of $\mathbb{R}^n$, and thus it is also an open cover of P in the subspace topology. Since P is compact there must be a finite subcover of $\mathcal{A}$ that covers P. This implies that there is a finite subcover of $\mathcal{C}$ that covers A. Thus A is compact. □

An immediate consequence of Theorem 6.42 is that S^n considered as a subspace of $\mathbb{R}^{n+1}$ is compact, and hence, that the torus $T^2 = S^1 \times S^1$ is compact.

We end this section with a proof of the generalized extreme value theorem.

Theorem 6.43 **(Generalized extreme value theorem)** *Let X be a compact space, and let $f: X \to \mathbb{R}$ be a continuous map where $\mathbb{R}$ is given the standard topology. Then there are $m, M \in X$ such that*

$$f(m) \leqslant f(x) \leqslant f(M)$$

for all $x \in X$.

Proof. By Theorem 6.37, $f(X)$ is compact. We must show that $f(X)$ contains its supremum and its infimum. If it does, then by setting $f(m) = \inf f(X)$ and $f(M) = \sup f(X)$ the theorem follows.

We prove that $f(X)$ contains its supremum. The proof for the infimum is similar. Since $f(X)$ is compact, it is closed and bounded by Theorem 6.42. In particular, $f(X)$ is bounded above. Hence, the set $f(X)$ has a least upper bound; $s = \sup f(X)$. Thus $p \leqslant s$ for all $p \in f(X)$.

We must show that $s \in f(X)$. Assume that $s \notin f(X)$. Since $f(X)$ is closed, i.e., $f(X)^c = \mathbb{R} \setminus f(X)$ is open, it follows that there is a real number $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subseteq f(X)^c$, and so,

$$(s - \epsilon, s + \epsilon) \cap f(X) = \emptyset.$$

Hence, there is a real number y such that y is an upper bound of $f(X)$ and $s - \epsilon < y < s$. This is a contradiction to the fact that s is the least upper bound of $f(X)$. Hence, $s = \sup f(X) \in f(X)$. □

6.4 Exercises

Exercise 6.1 Let X and Y be topological spaces. Show that if $f: X \to Y$ is surjective continuous map and X is path connected then Y is also path connected.

We say that a subset A of a topological space X is *path connected* if A is path connected in the subspace topology. Hence, the previous exercise may be extended to saying that the continuous image of a path connected space is path connected.

Exercise 6.2 Let X be a topological space, and let $A \subseteq B \subseteq \overline{A}$ be subspaces of X.

(a) Show that if A is connected then so is B.

(b) Show that $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b]$ and $[a, \infty)$ are all connected spaces when considered as subspaces of $\mathbb{R}$ with the standard topology.

Exercise 6.3 Let X be a topological space, and consider $I = [0, 1]$ as a subspace of $\mathbb{R}$ where $\mathbb{R}$ is given the standard topology. Furthermore, let the *cone on* X be quotient space $CX = X \times I / \sim$, where $\sim$ is the equivalence relation on the product space $X \times I$ given by $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Show that CX is path connected.

Exercise 6.4 Let X be a Hausdorff space, and let A be a subspace of X. Show that A is Hausdorff.

Exercise 6.5 Let X be an infinite set with the cofinite topology.

(a) Show that X is compact.

(b) Show that any subset of X is compact.

Exercise 6.6 Let X be a compact space, and let Y be a Hausdorff space. Furthermore, let $f\colon X \to Y$ be a continuous map.

(a) Show that f is a closed map.

(b) Show that if f is a surjective continuous map, then f is a quotient map.

(c) Show that if f a bijective continuous map, then f is a homeomorphism.

(d) Show that f is *proper*, i.e., for each subset K of Y that is compact the preimage $f^{-1}(K)$ is compact.

Exercise 6.7 Show that the surface of the cube centered at the origin,

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid \max\{|x|, |y|, |z|\} = 1\},$$

and the 2-sphere,

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \sqrt{x^2 + y^2 + z^2} = 1\right\},$$

are homeomorphic where they are both considered to be subspaces of $\mathbb{R}^3$ with the standard topology.

Exercise 6.8 Let X be a topological space, and let $A_1.A_2, \ldots, A_n$ be subspaces of X each of which is compact. Show that

$$\bigcup_{i=1}^{n} A_i$$

is compact.

7. The fundamental group

7.1 Homotopy of paths

The topological properties we have discussed so far do not help us to distinguish between, e.g., the 2-sphere S^2 and the torus $T^2 = S^1 \times S^1$. In order to prove that S^2 and T^2 are not topologically equivalent, i.e., $S^2 \ncong T^2$, we need new properties and new techniques. A such property is that of simple connectedness.

A path connected space X is, roughly speaking, *simply connected* if every closed curve in X can be shrunk to a point in X. The 2-sphere S^2 is simply connected as any closed curve on S^2 can be shrunk to a point in S^2. On the other hand, the torus T^2 is not simply connected as there are closed curves which cannot be shrunk to a point in T^2. See Figure 7.1. To be simply connected is a topological property. Hence, if X and Y are homeomorphic topological spaces and one of them is simply connected then so is the other. We can express that a path connected space is simply connected by saying that its *fundamental group* is trivial. In particular, if $X \cong Y$ then their fundamental groups are *isomorphic*. We will show that the fundamental group of S^2 is not isomorphic to the fundamental group of T^2. Hence, $S^2 \ncong T^2$. Computing fundamental groups provides us with more information than just that of being simply connected.

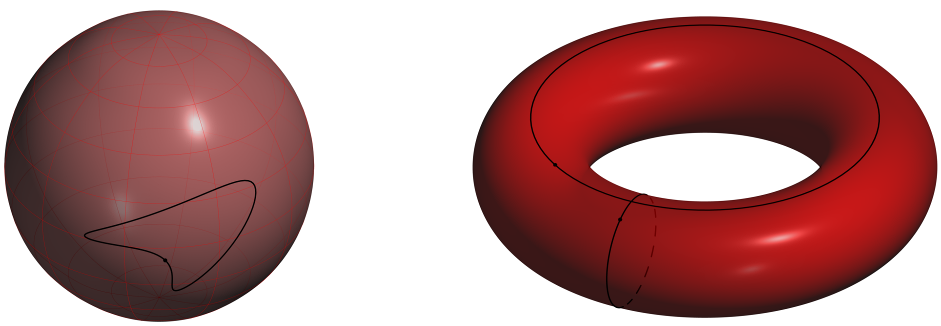


Figure 7.1: The 2-sphere S^2 is simply connected while the torus $T^2 = S^1 \times S^1$ is not simply connected.

In order to define the fundamental group of a topological space, we will need the concept of homotopy (of paths).

Definition 7.1 **(Homotopy)** Let X and Y be two topological spaces, and let $f_0, f_1 \colon X \to Y$ be two continuous maps. Furthermore, let $\mathbb{R}$ be the set of real numbers with the standard topology, $I = [0, 1]$ be a subspace of $\mathbb{R}$, and let $X \times I$ be given the product topology. We say that f_0 is *homotopic* to f_1, written $f_0 \simeq f_1$, if there is a continuous map

$$H \colon X \times I \to Y$$

such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. The map H is called a *homotopy* between f_0 and f_1. If $f_0 \simeq f_1$ and f_1 is a constant map, we say that f_0 is *nullhomotopic*.

We can think of a homotopy as a continuous one-parameter family $\{H|_{X\times\{t\}}\}_{t\in I}$ of continuous maps from X to Y. See Figure 7.2.

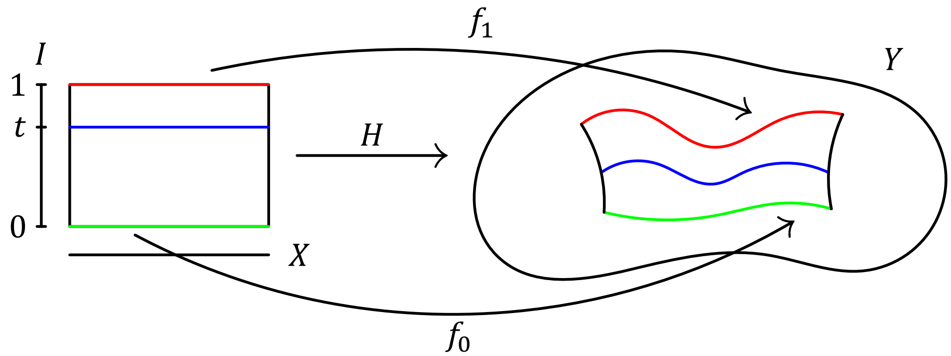


Figure 7.2: A homotopy from f_0 to f_1.

Example 7.2 (Straight-line homotopy) Let X be a topological space. Then any two continuous maps $f_0, f_1 \colon X \to \mathbb{R}^n$, where $\mathbb{R}^n$ is given the standard topology, are homotopic. The map $H \colon X \times I \to \mathbb{R}^n$ given by

$$H(x,t) = (1-t)f_0(x) + tf_1(x).$$

for all $x \in X$ and all $t \in I = [0,1]$ is a homotopy between f_0 and f_1. We call it the *straight-line homotopy* as it deforms for each $x \in X$ the point $f_0(x)$ to the point $f_1(x)$ along the straight-line segment between them.

In particular, any continuous map $f \colon X \to \mathbb{R}^n$ is homotopic to the constant map $c \colon X \to \mathbb{R}^n$ given by $c(x) = 0$ for $x \in X$. Hence, any continuous map into $\mathbb{R}^n$ is nullhomotopic.

We will need the following lemma to prove that $\simeq$ is an equivalence relation on the set of all continuous maps between a topological space X and a topological space Y.

Lemma 7.3 (Pasting lemma) *Let $X = A \cup B$ be a topological space, where A and B are closed in X. Furthermore, let Y be a topological space, and assume that $f \colon A \to Y$ and $g \colon B \to Y$ are continuous maps. If $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h \colon X \to Y$ given by*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. We will prove that the preimage of a closed subset of Y under h is closed in X. Let C be a closed subset of Y. Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous, it follows that $f^{-1}(C)$ is closed in A, and so, it is closed in X. Similarly, $g^{-1}(C)$ is closed in B, and so, it is closed in X. Hence,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

is closed in X as it is the union of two closed subsets of X. Thus h is continuous. □

Theorem 7.4 *The relation $\simeq$ is an equivalence relation on the set of all continuous maps from a topological space X to a topological space Y.*

Proof. We must show that $\simeq$ is reflexive, symmetric, and transitive. For any continuous map $f\colon X \to Y$ there is a homotopy $H\colon X \times I \to Y$ given by $H(x,t) = f(x)$ for all $x \in X$ and all $t \in I$. Then $H(x,0) = f(x)$ and $H(x,1) = f(x)$ for all $x \in X$. Hence, $f \simeq f$. Thus $\simeq$ is reflexive.

Let $f_0\colon X \to Y$ and $f_1\colon X \to Y$ be two continuous maps, and let $H\colon X \times I \to Y$ be a homotopy between them. Then $\overline{H}\colon X \times I \to Y$ given by $\overline{H}(x,t) = H(x,1-t)$ is a homotopy between f_1 and f_0. Hence, $f_0 \simeq f_1$ implies $f_1 \simeq f_0$. Thus $\simeq$ is symmetric.

To show that $\simeq$ is transitive, we must show that if $f_0 \simeq f_1$ and $f_1 \simeq f_2$, then $f_0 \simeq f_2$. Let $H_1\colon X \times I \to Y$ be a homotopy from f_0 to f_1, and let $H_2\colon X \times I \to Y$ be a homotopy from f_1 to f_2. Then there is a map $H_1 * H_2\colon X \times I \to Y$ given by

$$(H_1 * H_2)(x,t) = \begin{cases} H_1(x,2t) & 0 \leqslant t \leqslant 1/2 \\ H_2(x,2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

It is continuous for all $t \in I = [0,1]$ by Lemma 7.3 since $X \times [0,1/2]$ and $X \times [1/2,1]$ are closed subsets of $X \times I$ whose union is equal to $X \times I$, and $H_1(x,1) = H_2(x,0)$ for all $x \in X$. Hence, $H_1 * H_2$ is a homotopy from f_0 to f_2. Thus $\simeq$ is transitive. □

Definition 7.5 **(Homotopy classes)** Let X and Y be topological spaces, and let $C(X,Y)$ be the set of continuous maps from X to Y. The *homotopy classes* in $C(X,Y)$ are the equivalence classes under the relation $\simeq$. We write $[f]$ for the homotopy class of $f \in C(X,Y)$, i.e.,

$$[f] = \{g \in C(X,Y) \mid f \simeq g\},$$

and we write $[X,Y]$ for the set of homotopy classes of continuous maps from X to Y, i.e., $[X,Y] = C(X,Y)/\simeq$.

If we can connect two points x and y in a topological space X by a path, i.e., a continuous map

$$f\colon [a,b] \to X$$

such that $f(a) = x$ and $f(b) = y$, cf. Definition 6.14, we say that they lie in the same *path component*. Specifically, the relation $\sim_p$ given by $x \sim_p y$ if and only if there is a path from x to y in X, is an equivalence relation on X. We say that an equivalence class in X under $\sim_p$ is a *path component*. We write $\pi_0(X)$ for the set of path components of X. Note that $\pi_0(X)$ consists of only one element if and only if X is path connected.

Example 7.6 Let $X = \{x_0\}$ and Y be topological spaces. Then the continuous maps $f\colon X \to Y$ correspond to the points $f(x_0) = y \in Y$. Two continuous maps $f_0, f_1\colon X \to Y$ are homotopic if and only if there is a path from $y_0 = f_0(x_0)$ to $y_1 = f_1(x_0)$. Hence, the homotopy classes of continuous maps from X to Y correspond to the path components of Y, and $[X,Y]$ corresponds to $\pi_0(Y)$.

We are particularly interested in the case where we have homotopies between paths in a topological space X. For simplicity, let all paths be continuous maps from $I = [0,1]$ to X.

Definition 7.7 **(Path homotopy)** Let X be a topological space, and let $x_0, x_1 \in X$. We say that two paths $f, g\colon I \to X$ in X from x_0 to x_1 are *path homotopic*, written $f \simeq_p g$, if there is a continuous map $H\colon I \times I \to X$ such that

$$H(s,0) = f(s) \qquad \text{and} \qquad H(s,1) = g(s)$$

for all $s \in I$, and

$$H(0,t) = x_0 \qquad \text{and} \qquad H(1,t) = x_1$$

for all $t \in I$. We call H a *path homotopy* from f to g.

Example 7.8 Let $f\colon I \to \mathbb{R}^n$ and $g\colon I \to \mathbb{R}^n$ be paths from p_0 to p_1 in $\mathbb{R}^n$ where $\mathbb{R}^n$ is given the standard topology. Then they are path homotopic: the straight-line homotopy $H\colon I \times I \to \mathbb{R}^n$ given by

$$H(s,t) = (1-t)f(s) + tg(s)$$

for all $s \in I$ and all $t \in I$ is a path homotopy from f to g. See Figure 7.3.

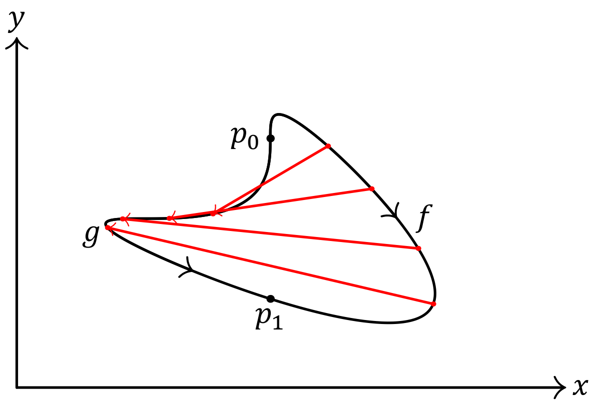


Figure 7.3: The straight-line homotopy between two paths in $\mathbb{R}^2$ is a path homotopy.

Example 7.9 Let $f\colon I \to \mathbb{R}^2 \setminus \{(0,0)\}$, $g\colon I \to \mathbb{R}^2 \setminus \{(0,0)\}$ and $h\colon I \to \mathbb{R}^2 \setminus \{(0,0)\}$ be paths from $(1,0)$ to $(-1,0)$ in $\mathbb{R}^2$, where $\mathbb{R}^2$ is given the standard topology, given by

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$g(s) = \left(\cos(\pi s), \frac{1}{2}\sin(\pi s)\right)$$

and

$$h(s) = (\cos(\pi s), -\sin(\pi s))$$

for all $s \in I$. Then the straight-line homotopy from f to g is a path homotopy but there are no path homotopies from either f or g to h. See Figure 7.4.

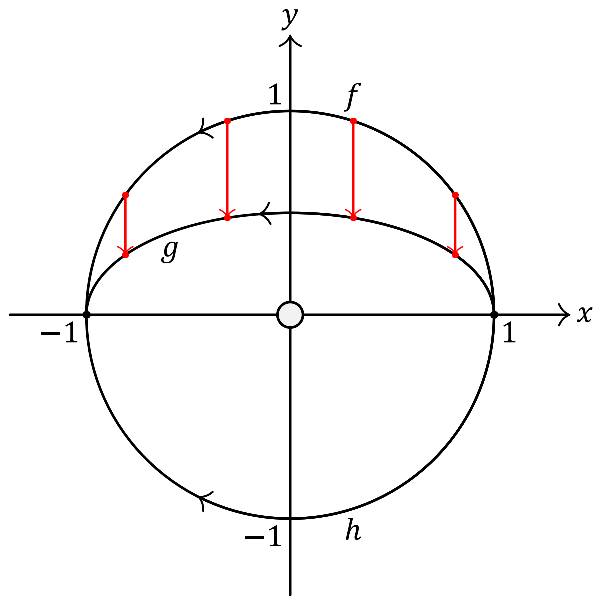


Figure 7.4: There are no path homotopies between the path homotopic paths f and g, and h in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Theorem 7.10 *Let X be a topological space, and let $x_0, x_1 \in X$. Then the relation $\simeq_p$ is an equivalence relation on the set of all paths from x_0 to x_1 in X.*

Proof. The result follows by proving that the homotopies constructed in the proof of Theorem 7.4 when applied to paths are path homotopies.

Let $f: I \to X$ be a path from x_0 to x_1. Then the constant homotopy $H: I \times I \to X$ given by

$$H(s,t) = f(s)$$

is path homotopy from f to f: $H(s,0) = f(s)$ and $H(s,1) = f(s)$ for all $s \in I$, and $H(0,t) = f(0) = x_0$ and $H(1,t) = f(1) = x_1$ for all $t \in I$.

Assume that $H: I \times I \to X$ is a path homotopy between paths $f: I \to X$ and $g: I \to X$ from x_0 to x_1. Then the homotopy $\overline{H}: I \times I \to X$ given by

$$\overline{H}(s,t) = H(s, 1-t)$$

is a path homotopy from g to f: $\overline{H}(s,0) = H(s,1) = g(s)$ and $\overline{H}(s,1) = H(s,0) = f(s)$ for all $s \in I$, and $\overline{H}(0,t) = H(0,1-t) = x_0$ and $\overline{H}(1,t) = H(1,1-t) = x_1$ for all $t \in I$.

Finally, assume that $f_0: I \to X$, $f_1: I \to X$ and $f_2: I \to X$ are paths from x_0 to x_1, and that $H_1: I \times I \to X$ is a path homotopy from f_0 to f_1 and that $H_2: I \times I \to X$ is a path homotopy from f_1 to f_2. Then the homotopy $H_1 * H_2: I \times I \to X$ given by

$$(H_1 * H_2)(s,t) = \begin{cases} H_1(s,2t) & 0 \leqslant t \leqslant 1/2 \\ H_2(s,2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

is a path homotopy from f_0 to f_2: $(H_1 * H_2)(s,0) = H_1(s,0) = f_0(s)$ and $(H_1 * H_2)(s,1) = H_2(s,1) = f_2(s)$ for all $s \in I$, and $(H_1 * H_2)(0,t) = x_0$ and $(H_1 * H_2)(1,t) = x_1$ for all $t \in I$. □

Definition 7.11 **(Path homotopy classes)** Let X be a topological space, and let $x_0, x_1 \in X$. If $f\colon I \to X$ is a path from x_0 to x_1, we write $[f]$ for its *path homotopy class*, i.e.,

$$[f] = \{g\colon I \to X \mid g \text{ is a path from } x_0 \text{ to } x_1 \text{ and } f \simeq_p g\}.$$

The fundamental group of a topological space X with base point x_0 is, as a set, the set of path homotopy classes of paths from x_0 to x_0, i.e., the set of path homotopy classes of *loops* based at x_0. The group structure of the fundamental group is derived from the following product.

Definition 7.12 **(Product of paths)** Let X be a topological space, and let $x_0, x_1, x_2 \in X$. If $f\colon I \to X$ is a path from x_0 to x_1, and $g\colon I \to X$ is a path from x_1 to x_2, we define the *product* of f and g as the path $f * g\colon I \to X$ from x_0 to x_2 given by

$$(f * g)(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2 \\ g(2s-1) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Note that the map $f * g$ is well-defined and continuous by Lemma 7.3. Furthermore, as $(f * g)(0) = f(0) = x_0$ and $(f * g)(1) = g(1) = x_2$, $f * g$ is a path from x_0 to x_2. See Figure 7.5.

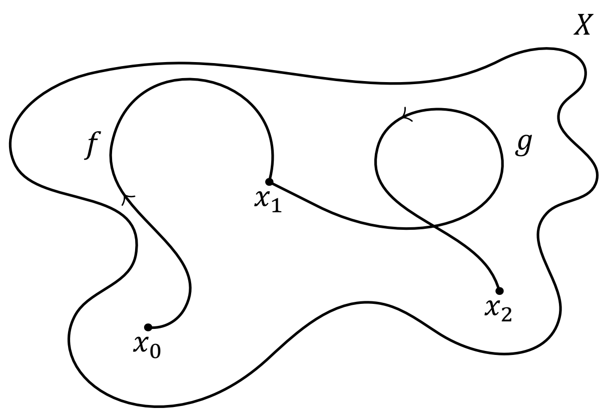


Figure 7.5: The product $f * g$.

Lemma 7.13 *Let X be a topological space, and let $x_0, x_1, x_2 \in X$. If $f\colon I \to X$ is a path from x_0 to x_1 and $g\colon I \to X$ is a path from x_1 to x_2, then the product $f * g$ induces a well-defined operation on path homotopy classes given by*

$$[f] * [g] = [f * g].$$

Proof. Let $f'\colon I \to X$ be a path from x_0 to x_1, and let $g'\colon I \to X$ be a path from x_1 to x_2. Furthermore, let $H_1\colon I \times I \to X$ be a path homotopy from f to f', and let $H_2\colon I \times I \to X$ be a path homotopy from g to g'. Then the map $H_1 * H_2\colon I \times I \to X$ given by

$$(H_1 * H_2)(s,t) = \begin{cases} H_1(2s,t) & 0 \leqslant s \leqslant 1/2 \\ H_2(2s-1,t) & 1/2 \leqslant s \leqslant 1 \end{cases}$$

is a path homotopy from $f * g$ to $f' * g'$: it is well-defined since $H_1(1,t) = x_1 = H_2(0,t)$ for all $t \in I$ and it is continuous by Lemma 7.3, and

$$\begin{aligned} &(H_1 * H_2)(s,0) = (f * g)(s) && \text{and} && (H_1 * H_2)(s,1) = (f' * g')(s) && \text{for all } s \in I, \\ &(H_1 * H_2)(0,t) = H_1(0,t) = x_0 && \text{and} && (H_1 * H_2)(1,t) = H_2(1,t) = x_2 && \text{for all } t \in I. \end{aligned}$$

Hence, $f * g$ induces a well-defined operation on path homotopy classes given by $[f] * [g] = [f * g]$. □

Theorem 7.14 *Let X be a topological space. Then the product of paths, $*$, has the following properties on the set of path homotopy classes in X.*

Associativity *Let x_0, x_1, x_2 and x_3 be points in X. If $f_0 \colon I \to X$ is a path from x_0 to x_1, $f_1 \colon I \to X$ is a path from x_1 to x_2, and $f_2 \colon I \to X$ is a path from x_2 to x_3, then*

$$([f_0] * [f_1]) * [f_2] = [f_0] * ([f_1] * [f_2]).$$

Left and right units *For $x \in X$, let $c_x \colon I \to X$ denote the* constant path *at x, given by $c_x(s) = x$ for all $s \in I$. If $f \colon I \to X$ is a path from x_0 to x_1 then*

$$[c_{x_0}] * [f] = [f] = [f] * [c_{x_1}].$$

Inverse *If $f \colon I \to X$ is a path from x_0 to x_1, let $\overline{f} \colon I \to X$ be the* reverse path *from x_1 to x_0, given by $\overline{f}(s) = f(1-s)$ for all $s \in I$. Then*

$$[f] * [\overline{f}] = [c_{x_0}] \qquad \text{and} \qquad [\overline{f}] * [f] = [c_{x_1}].$$

Proof. We prove that $*$ is an associative operation on the set of path homotopy classes in X. By definition, we have

$$\begin{aligned} ((f_0 * f_1) * f_2)(s) &= \begin{cases} (f_0 * f_1)(2s) & 0 \leqslant s \leqslant 1/2 \\ f_2(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \begin{cases} f_0(4s) & 0 \leqslant s \leqslant 1/4 \\ f_1(4s-1) & 1/4 \leqslant s \leqslant 1/2 \\ f_2(2s-1) & 1/2 \leqslant s \leqslant 1, \end{cases} \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} (f_0 * (f_1 * f_2))(s) &= \begin{cases} f_0(2s) & 0 \leqslant s \leqslant 1/2 \\ (f_1 * f_2)(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \begin{cases} f_0(2s) & 0 \leqslant s \leqslant 1/2 \\ f_1(4s-2) & 1/2 \leqslant s \leqslant 3/4 \\ f_2(4s-3) & 3/4 \leqslant s \leqslant 1. \end{cases} \end{aligned} \tag{7.2}$$

We must show that there is a path homotopy $H \colon I \times I \to X$ from $(f_0 * f_1) * f_2$ to $f_0 * (f_1 * f_2)$. If there is such a path homotopy then $[(f_0 * f_1) * f_2] = [f_0 * (f_1 * f_2)]$, and hence, by Lemma 7.13 $([f_0] * [f_1]) * [f_2] = [f_0] * ([f_1] * [f_2])$.

We can construct a path homotopy $H\colon I\times I\to X$ from $(f_0*f_1)*f_2$ to $f_0*(f_1*f_2)$ as follows. Draw the straight lines connecting the domain of f_1 in Equation 7.1 and the domain of f_1 in Equation 7.2 in $I\times I$. See Figure 7.6.

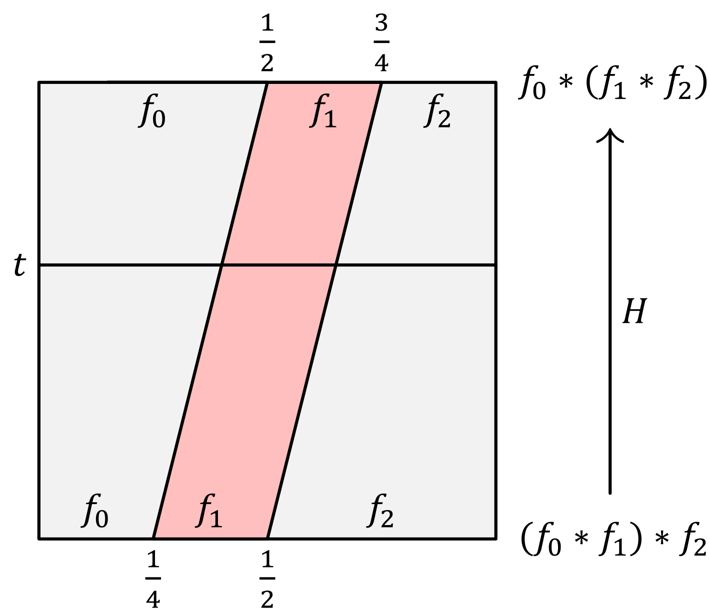


Figure 7.6: The path homotopy $H\colon I\times I\to X$ from $(f_0*f_1)*f_2$ to $f_0*(f_1*f_2)$.

The equation for the straight line from $(s,t)=(1/4,0)$ to $(s,t)=(1/2,1)$ is

$$s=\frac{1+t}{4},$$

and the equation for the straight line from $(s,t)=(1/2,0)$ to $(s,t)=(3/4,1)$ is

$$s=\frac{2+t}{4}.$$

We may then define $H\colon I\times I\to X$ by

$$H(s,t)=\begin{cases} f_0(4s/(1+t)) & 0\leqslant s\leqslant (1+t)/4\\ f_1(4s-t-1) & (1+t)/4\leqslant s\leqslant (2+t)/4\\ f_2((4s-t-2)/(2-t)) & (2+t)/4\leqslant s\leqslant 1.\end{cases}$$

By Lemma 7.3, H is continuous. Since

$$H(s,0)=\begin{cases} f_0(4s) & 0\leqslant s\leqslant 1/4\\ f_1(4s-1) & 1/4\leqslant s\leqslant 1/2\\ f_2(2s-1) & 1/2\leqslant s\leqslant 1\end{cases}$$

it follows that $H(s,0)=((f_0*f_1)*f_2)(s)$ for all $s\in I$ by Equation 7.1. Similarly, since

$$H(s,1)=\begin{cases} f_0(2s) & 0\leqslant s\leqslant 1/2\\ f_1(4s-2) & 1/2\leqslant s\leqslant 3/4\\ f_2(4s-3) & 3/4\leqslant s\leqslant 1\end{cases}$$

it follows that $H(s,1) = (f_0 * (f_1 * f_2))(s)$ for all $s \in I$ by Equation 7.2. Finally, since $H(0,t) = f_0(0) = x_0$ and $H(1,t) = f_2(1) = x_3$ for all $t \in I$, it follows that H is a path homotopy from $(f_0 * f_1) * f_2$ to $f_0 * (f_1 * f_2)$.

We prove that the constant path is a left unit under $*$ for path homotopy classes. The proof for the fact that the constant path is a right unit under $*$ is similar. By definition, we have

$$(c_{x_0} * f)(s) = \begin{cases} c_{x_0}(2s) & 0 \leqslant s \leqslant 1/2 \\ f(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases}$$
$$= \begin{cases} x_0 & 0 \leqslant s \leqslant 1/2 \\ f(2s-1) & 1/2 \leqslant s \leqslant 1. \end{cases} \tag{7.3}$$

We must show that there is a path homotopy $H'\colon I \times I \to X$ from $c_{x_0} * f$ to f. If there is such a path homotopy then $[c_{x_0} * f] = [f]$, and hence, by Lemma 7.13 $[c_{x_0}] * [f] = [f]$.

We can construct a path homotopy $H'\colon I \times I \to X$ from $c_{x_0} * f$ to f as follows. Draw the straight line from $(s,t) = (1/2, 0)$ to $(s,t) = (0,1)$. See Figure 7.7.

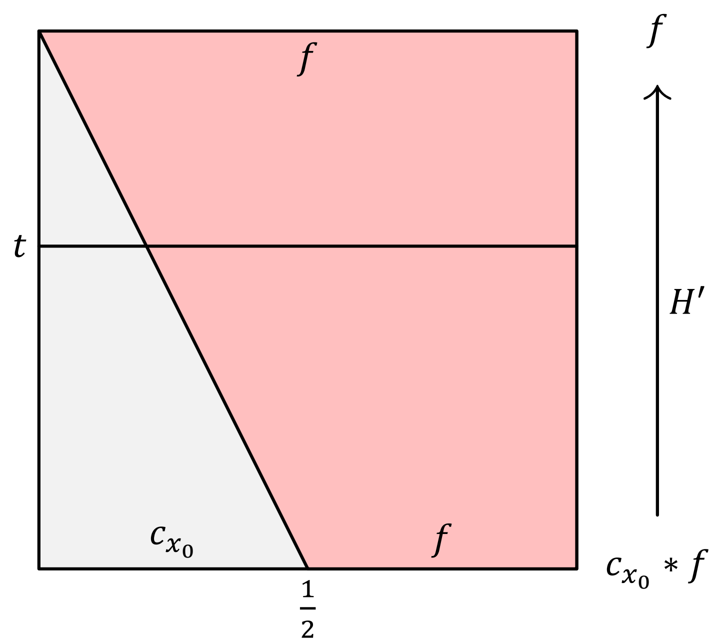


Figure 7.7: The path homotopy $H'\colon I \times I \to X$ from $c_{x_0} * f$ to f.

The equation for the straight line from $(s,t) = (1/2,0)$ to $(s,t) = (0,1)$ is given by

$$s = \frac{1-t}{2}.$$

We may then define $H'\colon I \times I \to X$ by

$$H'(s,t) = \begin{cases} x_0 & 0 \leqslant s \leqslant (1-t)/2 \\ f((2s+t-1)/(1+t)) & (1-t)/2 \leqslant s \leqslant 1. \end{cases}$$

By Lemma 7.3, H' is continuous. Since

$$H'(s,0) = \begin{cases} x_0 & 0 \leqslant s \leqslant 1/2 \\ f(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases}$$

it follows by Equation 7.3 that $H'(s,0) = (c_{x_0} * f)(s)$, and clearly, $H'(s,1) = f(s)$, for all $s \in I$. Finally, since $H'(0,t) = x_0$ and $H'(1,t) = x_1$ for all $t \in I$, it follows that H' is a path homotopy from $c_{x_0} * f$ to f.

Finally, we prove that for each path homotopy class there is an inverse path homotopy class under $*$. By definition, we have

$$\begin{aligned}(f * \overline{f})(s) &= \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2 \\ \overline{f}(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2 \\ f(1-(2s-1)) & 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2 \\ f(2-2s) & 1/2 \leqslant s \leqslant 1. \end{cases}\end{aligned} \tag{7.4}$$

Hence, $f * \overline{f}$ is a path from x_0 to x_0. We must show that there is a path homotopy $H'' \colon I \times I \to X$ from $f * \overline{f}$ to c_{x_0}.

We can construct such a path homotopy as follows. Draw a straight line from $(s,t) = (0,0)$ to $(s,t) = (1/2,1)$ and a straight line from $(s,t) = (1,0)$ to $(s,t) = (1/2,1)$. See Figure 7.8.

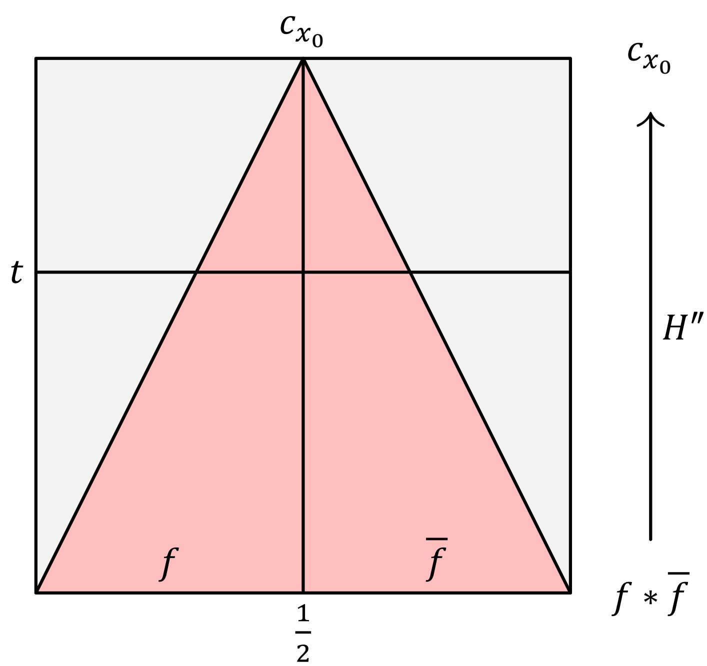


Figure 7.8: The path homotopy $H'' \colon I \times I \to X$ from $f * \overline{f}$ to c_{x_0}.

The equation for the straight line from $(s,t) = (0,0)$ to $(s,t) = (1/2,1)$ is given by

$$s = \frac{t}{2},$$

and the equation for the straight line from $(s,t) = (1,0)$ to $(s,t) = (1/2,1)$ is given by

$$s = 1 - \frac{t}{2}.$$

We may then define $H'' : I \times I \to X$ by

$$H''(s,t) = \begin{cases} x_0 & 0 \leqslant s \leqslant t/2 \\ f(2s-t) & t/2 \leqslant s \leqslant 1/2 \\ f(2-2s-t) & 1/2 \leqslant s \leqslant 1-t/2 \\ x_0 & 1-t/2 \leqslant s \leqslant 1. \end{cases}$$

By Lemma 7.3, H'' is continuous. Since

$$H''(s,0) = \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2 \\ f(2-2s) & 1/2 \leqslant s \leqslant 1 \end{cases}$$

it follows by Equation 7.4 that $H''(s,0) = (f * \overline{f})(s)$, and $H''(s,1) = x_0 = c_{x_0}(s)$ for all $s \in I$. Finally, since $H''(0,t) = x_0$ and $H''(1,t) = x_0$ for all $t \in I$, it follows that H'' is a path homotopy from $f * \overline{f}$ to c_{x_0}. The construction of a path homotopy from $\overline{f} * f$ to c_{x_1} is similar. □

7.2 Definition and elementary properties of the fundamental group

The set of path homotopy classes of paths in a topological space X does not form a group under the product of paths as the product of two path homotopy classes is not always defined. However, such a product is defined if we restrict to paths that begin and end at some fixed point x_0. This is the fundamental group of X with basepoint x_0. We say that the ordered pair (X, x_0) is a *based space* (sometimes also called a *pointed space*).

Definition 7.15 (The fundamental group) Let (X, x_0) be a based space. A path $f : I \to X$ from x_0 to x_0 is called a *loop in X based at x_0*. Let

$$\pi_1(X, x_0) = \{[f] \mid f \text{ is a loop in } X \text{ based at } x_0\}$$

be the set of path homotopy classes of loops in X based at x_0. We say that $\pi_1(X, x_0)$ is the *fundamental group* of X based at x_0.

From the definition it is clear that $\pi_1(X, x_0)$ depends only on the path component of X that contains x_0. The fundamental group does not provide us with any information about the rest of X.

As the notation suggests, the fundamental group is sometimes also referred to as the *first homotopy group* of X. In fact, there are groups $\pi_n(X, x_0)$ for all $n \in \mathbb{Z}_+$. These groups are part of a subject called *homotopy theory*.

If we restrict Theorem 7.14 to the case where all paths are loops in X based at x_0, we get the following result.

Theorem 7.16 *Let (X, x_0) be a based space. Then the fundamental group $\pi_1(X, x_0)$ of X based at x_0 is, in fact, a group with product of paths, $*$, as its binary operation. The identity element e is equal to the path homotopy class of the constant path at x_0, $e = [c_{x_0}]$, and the inverse of $[f]$ is $[f]^{-1} = [\overline{f}]$ where $\overline{f}$ is the reverse path of f.*

Example 7.17 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. If $f: I \to \mathbb{R}$ is a loop in $\mathbb{R}$ based at 0, by Example 7.2 we know that f is path homotopic to the constant path at 0. Thus $\pi_1(\mathbb{R}, 0)$ is the trivial group, i.e., $\pi_1(\mathbb{R}, 0) = \{e\}$. In general, $\pi_1(\mathbb{R}^n, 0)$ is the trivial group, where $\mathbb{R}^n$ is given the standard topology. We will often denote the trivial group as simply 0.

Example 7.18 Let (X, x_0) be a based space where X is a discrete space. Then $\pi_1(X, x_0) = 0$ for all $x_0 \in X$ as the path component containing x_0 consists of only one element, namely x_0.

Theorem 7.19 *Let X be a path connected space, and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Proof. Let $\alpha: I \to X$ be a path from x_0 to x_1. If $f: I \to X$ is a loop in X based x_0, then $\overline{\alpha} * f * \alpha$ is a loop in X based at x_1. We define the map $\widehat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\widehat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha]$$

where we have used the fact that $[\overline{\alpha}] = [\alpha]^{-1}$. It is a well-defined map as $*$ is well-defined, and moreover, it only depends on the path homotopy class of α.

We will show that $\widehat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism. We will first show that $\widehat{\alpha}$ is a homomorphism, and then show that it is bijective. Since

$$\begin{aligned}\widehat{\alpha}([f] * [g]) &= [\alpha]^{-1} * [f] * [g] * [\alpha] \\ &= ([\alpha]^{-1} * [f] * [\alpha]) * ([\alpha]^{-1} * [g] * [\alpha]) \\ &= \widehat{\alpha}([f]) * \widehat{\alpha}([g]),\end{aligned}$$

for all $[f], [g] \in \pi_1(X, x_0)$, it follows that $\widehat{\alpha}$ is a homomorphism. To show that $\widehat{\alpha}$ is bijective, and hence, an isomorphism, we show that $\widehat{\alpha}$ has an inverse. Let $\beta = \overline{\alpha}$, i.e., β is the reverse path of α. Since

$$\begin{aligned}\widehat{\beta}([h]) &= [\beta]^{-1} * [h] * [\beta] = [\alpha] * [h] * [\alpha]^{-1} \\ \widehat{\alpha}(\widehat{\beta}([h])) &= [\alpha]^{-1} * ([\beta]^{-1} * [h] * [\beta]) * [\alpha] = [h]\end{aligned}$$

for all $[h] \in \pi_1(X, x_1)$, and

$$\begin{aligned}\widehat{\alpha}([f]) &= [\alpha]^{-1} * [f] * [\alpha] = [\beta] * [f] * [\beta]^{-1} \\ \widehat{\beta}(\widehat{\alpha}([f])) &= [\beta]^{-1} * ([\alpha]^{-1} * [f] * [\alpha]) * [\beta] = [f]\end{aligned}$$

for all $[f] \in \pi_1(X, x_0)$, it follows that $\widehat{\beta} = (\widehat{\alpha})^{-1}$. □

In light of Theorem 7.19 it might be tempting to simply speak of the fundamental group of a path connected space X, and thus leave out any mention of the basepoint. However, while $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any two points x_0 and x_1 in X, the isomorphism may depend on the path from x_0 to x_1. It can be shown that the isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ for a path connected space X is independent of path if and only if the fundamental group is abelian.

Definition 7.20 (Simply connected spaces) Let X be a path connected space. We say that X is *simply connected* if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence, for all $x_0 \in X$.

By Example 7.17, we get that $\mathbb{R}^n$ is simply connected where $\mathbb{R}^n$ is given the standard topology. We shall later see that the circle S^1 and the torus $T^2 = S^1 \times S^1$ are two examples of non-simply connected spaces.

Definition 7.21 (Based maps) Let (X, x_0) and (Y, y_0) be based spaces. A *based map* $h\colon (X, x_0) \to (Y, y_0)$ is a continuous map $h\colon X \to Y$ such that $h(x_0) = y_0$.

Let (X, x_0) and (Y, y_0) be based spaces, and let $h\colon (X, x_0) \to (Y, y_0)$ be a based map. If $f : I \to X$ is a loop in X based at x_0, the composition $h \circ f : I \to Y$ is a loop in Y based at y_0. This leads to a map from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.

Definition 7.22 (Homomorphisms induced by based maps) Let (X, x_0) and (Y, y_0) be based spaces, and let $h\colon (X, x_0) \to (Y, y_0)$ be a based map. The map

$$h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by

$$h_*([f]) = [h \circ f]$$

is called the *homomorphism induced by h*.

Lemma 7.23 *Let (X, x_0) and (Y, y_0) be based spaces, and let $h\colon (X, x_0) \to (Y, y_0)$ be a based map. The map*

$$h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by

$$h_*([f]) = [h \circ f]$$

is a homomorphism.

Proof. We need to show that for two loops $f\colon I \to X$ and $g\colon I \to X$ in X based at x_0, the identity

$$h_*([f] * [g]) = h_*([f]) * h_*([g])$$

holds in $\pi_1(Y, y_0)$. The left-hand side is the path homotopy class of $h \circ (f * g)$ and the right-hand side is the path homotopy class of $(h \circ f) * (h \circ g)$. By definition, we have

$$((h \circ f) * (h \circ g))(s) = \begin{cases} (h \circ f)(2s) & 0 \leqslant s \leqslant 1/2 \\ (h \circ g)(2s - 1) & 1/2 \leqslant s \leqslant 1 \end{cases}$$

and so

$$(h \circ (f * g))(s) = ((h \circ f) * (h \circ g))(s) = \begin{cases} h(f(2s)) & 0 \leqslant s \leqslant 1/2 \\ h(g(2s - 1)) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Hence, $h \circ (f * g) = (h \circ f) * (h \circ g)$. Thus $h_*([f] * [g]) = h_*([f]) * h_*([g])$ holds in $\pi_1(Y, y_0)$. □

Theorem 7.24 **(Functoriality)** *Let (X, x_0), (Y, y_0) and (Z, z_0) be based spaces, and let $h_1 : (X, x_0) \to (Y, y_0)$ and $h_2 : (Y, y_0) \to (Z, z_0)$ be based maps. Then*

$$(h_2 \circ h_1)_* = (h_2)_* \circ (h_1)_*.$$

If $\mathrm{id}_X : X \to X$ is the identity map, then $(\mathrm{id}_X)_*$ is the identity automorphism of $\pi_1(X, x_0)$.

The rule sending a based space (X, x_0) to its fundamental group $\pi_1(X, x_0)$ and a based map $h : (X, x_0) \to (Y, y_0)$ to its induced homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a *functor* from the category of based spaces and based maps to the category of groups and (group) homomorphisms.

Proof. Assume that $f : I \to X$ is a loop in X based at x_0. Then

$$(h_2 \circ h_1)_*([f]) = [(h_2 \circ h_1) \circ f]$$

and

$$(h_2)_*\big((h_1)_*([f])\big) = [h_2 \circ (h_1 \circ f)].$$

Since composition of continuous maps is associative, i.e., $(h_2 \circ h_1) \circ f = h_2 \circ (h_1 \circ f)$, these path homotopy classes must be equal. Hence, $(h_2 \circ h_1)_* = (h_2)_* \circ (h_1)_*$.

If $\mathrm{id}_X : X \to X$ is the identity map, then $\mathrm{id}_X \circ f = f$, and so, $(\mathrm{id}_X)_*([f]) = [\mathrm{id}_X \circ f] = [f]$. Hence, $(\mathrm{id}_X)_*$ is the identity automorphism of $\pi_1(X, x_0)$. □

The following corollary of Theorem 7.24 tells us that the fundamental group is a topological invariant of based spaces.

Corollary 7.25 *Let (X, x_0) and (Y, y_0) be based spaces. If $h : X \to Y$ is a homeomorphism such that $h(x_0) = y_0$, then*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

The fundamental group thus allows us to distinguish between two based spaces provided that their fundamental groups are *not* isomorphic. By Example 7.17, we know that $\pi_1(\mathbb{R}, 0) = 0$ and $\pi_1(\mathbb{R}^2, 0) = 0$. However, by Example 6.16, we know that $\mathbb{R}$ and $\mathbb{R}^2$ are *not* homeomorphic. Hence, it is possible for two non-homeomorphic based spaces to have isomorphic fundamental groups.

Proof. We must show that h_* is bijective. Let $k : Y \to X$ be the inverse homemorphism of h such that $k(y_0) = x_0$. Then, by Theorem 7.24, $k_* \circ h_* = (k \circ h)_* = (\mathrm{id}_X)_*$, and $h_* \circ k_* = (h \circ k)_* = (\mathrm{id}_Y)_*$. Hence, $k_* = (h_*)^{-1}$. □

To compute the fundamental group of a product space, we may use the following theorem.

Theorem 7.26 *Let (X, x_0) and (Y, y_0) be based spaces. Then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ be the projection maps onto the first and second factor, respectively. Define the map $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$\varphi([h]) = \big((\mathrm{pr}_1)_*([h]), (\mathrm{pr}_2)_*([h])\big) = \big([\mathrm{pr}_1 \circ h], [\mathrm{pr}_2 \circ h]\big).$$

Note that $\pi_1(X, x_0) \times \pi_1(Y, y_o)$ is a group with binary operation given by

$$([f], [g]) * ([f'], [g']) = ([f] * [f'], [g] * [g'])$$

for $[f], [f'] \in \pi_1(X, x_0)$ and $[g], [g'] \in \pi_1(Y, y_0)$.

We will show that φ is an isomorphism. It is a homomorphism since $(\mathrm{pr}_1)_*$ and $(\mathrm{pr}_2)_*$ are homomorphisms:

$$\begin{aligned}
\varphi([h] * [k]) &= \big((\mathrm{pr}_1)_*([h] * [k]), (\mathrm{pr}_2)_*([h] * [k])\big) \\
&= \big((\mathrm{pr}_1)_*([h]) * (\mathrm{pr}_1)_*([k]), (\mathrm{pr}_2)_*([h]) * (\mathrm{pr}_2)_*([k])\big) \\
&= \big((\mathrm{pr}_1)_*([h]), (\mathrm{pr}_2)_*([h])\big) * \big((\mathrm{pr}_1)_*([k]), (\mathrm{pr}_2)_*([k])\big) \\
&= \varphi([h]) * \varphi([k])
\end{aligned}$$

for $[h], [k] \in \pi_1(X \times Y, (x_0, y_0))$. Let $f : I \to X$ be a loop in X based at x_0, and let $g : I \to Y$ be a loop in Y based at y_0. Define the map $h : I \to X \times Y$ by

$$h(s) = (f(s), g(s)).$$

Then h is a loop in $X \times Y$ based at (x_0, y_0), and

$$\varphi([h]) = \big((\mathrm{pr}_1)_*([h]), (\mathrm{pr}_2)_*([h])\big) = \big([\mathrm{pr}_1 \circ h], [\mathrm{pr}_2 \circ h]\big) = ([f], [g]).$$

Hence, φ is surjective. To show that φ is injective, assume that $\varphi([h]) = \varphi([k])$ for $[h], [k] \in \pi_1(X \times Y, (x_0, y_0))$. Then there is a path homotopy $H_1 : I \times I \to X$ from $\mathrm{pr}_1 \circ h$ to $\mathrm{pr}_1 \circ k$ and a path homotopy $H_2 : I \times I \to Y$ from $\mathrm{pr}_2 \circ h$ to $\mathrm{pr}_2 \circ k$. Define the map $H : I \times I \to X \times Y$ by

$$H(s, t) = (H_1(s, t), H_2(s, t)).$$

Then H is a path homotopy from $h = (\mathrm{pr}_1 \circ h, \mathrm{pr}_2 \circ h)$ to $k = (\mathrm{pr}_1 \circ k, \mathrm{pr}_2 \circ k)$. Hence, $[h] = [k]$, and so, φ is injective. Thus φ is an isomorphism. □

7.3 Homotopy type

We will see in this section how we may obtain information about the fundamental group by way of *homotopy types*. This allows us to compute the fundamental group of a topological space by computing the fundamental group of some other topological space that is hopefully easier to work with.

Lemma 7.27 *Let (X, x_0) and (Y, y_0) be based spaces, and let $h : (X, x_0) \to (Y, y_0)$ and $k : (X, x_0) \to (Y, y_0)$ be based maps. If there is a homotopy $H : X \times I \to Y$ from h to k such that $H(x_0, t) = y_0$ for all $t \in I$, then the homomorphisms $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by h and k, respectively, are equal.*

Proof. If $f\colon I \to X$ is a loop in X based at x_0, then $H \circ (f \times \mathrm{id})\colon I \times I \to Y$ is a homotopy from $h \circ f$ to $k \circ f$ where $\mathrm{id}\colon I \to I$ is the identity map of I. Since

$$\begin{array}{llll} (H \circ (f \times \mathrm{id}))(s,0) = (h \circ f)(s) & \text{and} & (H \circ (f \times \mathrm{id}))(s,1) = (k \circ f)(s) & \text{for all } s \in I, \\ (H \circ (f \times \mathrm{id}))(0,t) = y_0 & \text{and} & (H \circ (f \times \mathrm{id}))(1,t) = y_0 & \text{for all } t \in I, \end{array}$$

$H \circ (f \times \mathrm{id})$ is a path homotopy from $h \circ f$ to $k \circ f$. Hence, $h_*([f]) = [h \circ f] = [k \circ f] = k_*([f])$. Thus $h_* = k_*$. □

Definition 7.28 (Retractions) Let X be a topological space, and let A be a subspace of X. We say that a continuous map $r\colon X \to A$ is a *retraction* of X onto A if $r(a) = a$ for each $a \in A$. If there is a retraction of X onto A, we say that A is a *retract* of X.

Example 7.29 The circle, S^1, is a retract of $\mathbb{R}^2 \setminus \{0\}$ where $\mathbb{R}^2$ is given the standard topology. Specifically, the continuous map $r\colon \mathbb{R}^2 \setminus \{0\} \to S^1$ given by

$$r(x) = \frac{x}{\|x\|}$$

where $\|x\| = \sqrt{x_1^2 + x_2^2}$, is a retraction of $\mathbb{R}^2 \setminus \{0\}$ onto S^1.

If $i\colon A \to X$ is the inclusion map, then we can express that $r\colon X \to A$ is a retraction of X onto A by the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X \\ & \underset{\mathrm{id}_A}{\searrow} & \downarrow r \\ & & A \end{array}$$

Hence, for a retraction r, we have $r \circ i = \mathrm{id}_A$ where $\mathrm{id}_A\colon A \to A$ is the identity map of A.

Lemma 7.30 *Let X be a topological space, and let A be a subspace of X. If $x_0 \in A$ and A is a retract of X, then the homomorphism $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map $i\colon A \to X$ is a monomorphism.*

Proof. Let $r\colon X \to A$ be a retraction of X onto A. Then, by Theorem 7.24 and the definition of retraction, we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(A, x_0) & \xrightarrow{\;i_*\;} & \pi_1(X, x_0) \\ & \underset{(\mathrm{id}_A)_*}{\searrow} & \downarrow r_* \\ & & \pi_1(A, x_0) \end{array}$$

where id_A is the identity map of A. By Theorem 7.24, we know that $(\mathrm{id}_A)_*$ is an isomorphism. Thus $r_* \circ i_*$ is an isomorphism, and hence, also bijective. This implies that i_* is injective. Hence, i_* is a monomorphism. □

The homomorphism induced by the inclusion of A into X is an isomorphism if A is a *deformation retract*.

Definition 7.31 (Deformation retracts) Let X be a topological space, and let A be a subspace of X. A homotopy

$$H: X \times I \to X$$

is called a *deformation retraction* of X onto A if $H(x,0) = x$ and $H(x,1) \in A$ for all $x \in X$, and $H(a,t) = a$ for all $a \in A$ and all $t \in I$. We say that A is a *deformation retract* of X.

The map $r: X \to A$ defined by $r(x) = H(x,1)$ is a retraction of X onto A. Thus H is a homotopy between the identity map of X and the map $i \circ r$ where $i: A \to X$ is the inclusion map. Note that some authors call what we have defined as a deformation retract, a *strong* deformation retract. In this terminology a homotopy $H: X \times I \to Y$ is a *deformation retract* of X onto A if $H(x,0) = x$ and $H(x,1) \in A$ for all $x \in X$, and $H(a,1) = a$ for all $a \in A$.

Every deformation retract is also a retract but the converse is not true in general.

Example 7.32 Let X be a topological space, and let $a \in X$. Then $A = \{a\}$ is retract of X with retraction $r: X \to A$ given by $r(x) = a$. If A is a deformation retract of X, then X must be path connected: the deformation retraction gives that there is a path from each $x \in X$ to a. However, there are path connected spaces that do not deformation retract onto a point.

Theorem 7.33 *Let X be a topological space, and let A be a subspace of X. If $x_0 \in A$ and A is a deformation retract of X, then the homomorphism $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map $i: A \to X$ is an isomorphism.*

Proof. Let $H: X \times I \to X$ be a deformation retraction of X onto A, and let a retraction $r: X \to A$ of X onto A be given by $r(x) = H(x,1)$. By Lemma 7.27 and Theorem 7.24, we have $i_* \circ r_* = (\mathrm{id}_X)_*$ where id_X is the identity map of X. Since $r \circ i = \mathrm{id}_A$, we have $r_* \circ i_* = (\mathrm{id}_A)_*$ where id_A is the identity map of A. Thus both $r_* \circ i_*$ and $i_* \circ r_*$ are isomorphisms. Hence, $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism with r_* as its inverse. □

We may generalize the notion of deformation retraction. This leads us to the notion of *homotopy equivalence*.

Definition 7.34 (Homotopy equivalences) Let X and Y be topological spaces. If $f: X \to Y$ and $g: Y \to X$ are continuous maps such that $g \circ f$ is homotopic to the identity map of X, id_X, and $f \circ g$ is homotopic to the identity map of Y, id_Y, we say that f and g are *homotopy equivalences*. We say that each of f and g is a *homotopy inverse* of the other.

Clearly, every homeomorphism is also a homotopy equivalence. The converse is not true in general.

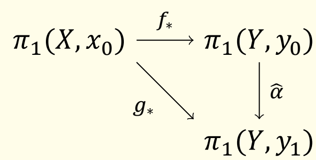
Example 7.35 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then for any $p \in \mathbb{R}$, the map $f: \mathbb{R} \to \{p\}$ given by $f(x) = p$ is a homotopy equivalence with homotopy inverse $g: \{p\} \to \mathbb{R}$ given by $g(p) = p$. It is clear that f is not a homeomorphism. In fact, there are no homeomorphisms between $\mathbb{R}$ and $\{p\}$.

Definition 7.36 (Homotopy types) Let X and Y be topological spaces. We say that X and Y have the same *homotopy type* if there is a homotopy equivalence $f\colon X \to Y$.

It is common to also refer to topological spaces of the same homotopy type as *homotopy equivalent*.

We end this section with a result showing that topological spaces of the same homotopy type have isomorphic fundamental groups. To prove the result we will need the following lemma.

Lemma 7.37 *Let X and Y be topological spaces, and let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps such that $f(x_0) = y_0$ and $g(x_0) = y_1$. If $H\colon X \times I \to Y$ is a homotopy from f to g, there is a path $\alpha\colon I \to Y$ in Y from y_0 to y_1 given by $\alpha(t) = H(x_0, t)$ such that $g_* = \widehat{\alpha} \circ f_*$.*



Proof. Let $h\colon I \to X$ be a loop in X based at x_0. We will show that

$$g_*([h]) = \widehat{\alpha}(f_*([h])).$$

In other words, we will show that

$$[g \circ h] = \widehat{\alpha}([f \circ h]) = [\alpha]^{-1} * [f \circ h] * [\alpha] = [\overline{\alpha} * (f \circ h) * \alpha].$$

Let $H'\colon I \times I \to Y$ be given by

$$H'(s,t) = \begin{cases} \overline{\alpha}(4s) & 0 \leqslant s \leqslant t/4 \\ H\left(h\left(\frac{4s-t}{4-3t}\right), 1-t\right) & t/4 \leqslant s \leqslant 1 - t/2 \\ \alpha(2s-1) & 1 - t/2 \leqslant s \leqslant 1. \end{cases}$$

See Figure 7.9.

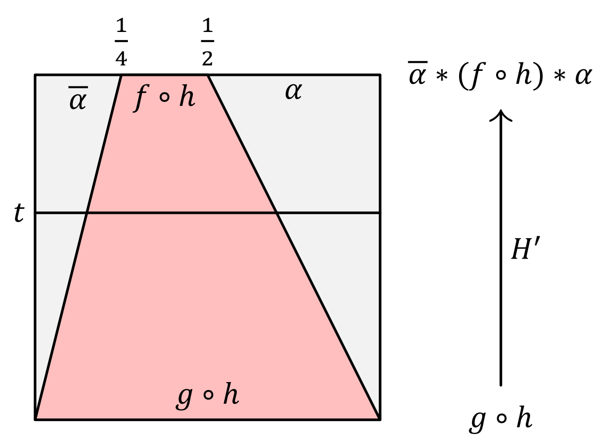


Figure 7.9: The path homotopy $H'\colon I \times I \to Y$ from $g \circ h$ to $\overline{\alpha} * (f \circ h) * \alpha$.

By Lemma 7.3, H' is continuous. Since

$$H'(s,0) = H(h(s),1) = g(h(s))$$

and

$$H'(s,1) = \begin{cases} \overline{\alpha}(4s) & 0 \leqslant s \leqslant 1/4 \\ H(h(4s-1),0) & 1/4 \leqslant s \leqslant 1/2 \\ \alpha(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases}$$
$$= \begin{cases} \overline{\alpha}(4s) & 0 \leqslant s \leqslant 1/4 \\ f(h(4s-1)) & 1/4 \leqslant s \leqslant 1/2 \\ \alpha(2s-1) & 1/2 \leqslant s \leqslant 1 \end{cases}$$

for all $s \in I$, and $H'(0,t) = H(x_0,1) = y_1$ and $H'(1,t) = H(x_0,1) = y_1$, it follows that H' is a path homotopy from $g \circ h$ to $\overline{\alpha} * (f \circ h) * \alpha$. Thus $[g \circ h] = [\overline{\alpha} * (f \circ h) * \alpha]$. This completes the proof. □

Theorem 7.38 *Let X and Y be topological spaces, and let $f\colon X \to Y$ be a homotopy equivalence such that $f(x_0) = y_0$. Then*

$$f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$$

is an isomorphism.

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f, and let $f(x_0) = y_0$, $g(y_0) = x_1$, and $f(x_1) = y_1$. By assumption, we have $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ where $\mathrm{id}_X\colon X \to X$ and $\mathrm{id}_Y\colon Y \to Y$ are the identity maps of X and Y, respectively. Thus, by Lemma 7.37, there is a path $\alpha\colon I \to X$ in X from x_0 to x_1 and a path $\beta\colon I \to Y$ from y_0 to y_1 such that

$$(g \circ (f_{x_0}))_* = g_* \circ (f_{x_0})_* = \widehat{\alpha} \circ (\mathrm{id}_X)_* = \widehat{\alpha}$$

and

$$((f_{x_1}) \circ g)_* = (f_{x_1})_* \circ g_* = \widehat{\beta} \circ (\mathrm{id}_Y)_* = \widehat{\beta}$$

where we have specified the basepoint for the homomorphisms induced by f. In other words, the following two diagrams commute.

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{(\mathrm{id}_X)_*} & \pi_1(X,x_0) \\ & {\scriptstyle g_* \circ (f_{x_0})_*} \searrow & \downarrow \widehat{\alpha} \\ & & \pi_1(X,x_1) \end{array} \qquad \begin{array}{ccc} \pi_1(Y,y_0) & \xrightarrow{g_*} & \pi_1(X,x_1) \\ & {\scriptstyle \widehat{\beta}} \searrow & \downarrow (f_{x_1})_* \\ & & \pi_1(Y,y_1) \end{array}$$

Since $\widehat{\alpha}$ and $\widehat{\beta}$ are both isomorphisms, it follows by commutativity of the diagram on the left that g_* is an epimorphism, and by commutativity of the diagram on the right that g_* is a monomorphism. Hence, g_* is an isomorphism. Since

$$(f_{x_0})_* = (g_*)^{-1} \circ \widehat{\alpha}$$

it follows that $(f_{x_0})_*$ is an isomorphism. □

We can use Theorem 7.38 to help us determine the fundamental group of certain topological spaces, and as a tool to determine that certain topological spaces are not of the same homotopy type, and hence, not homeomorphic.

7.4 Exercises

Exercise 7.1 Let X, Y and Z be topological spaces. Show that if $f: X \to Y$ is homotopic to $f': X \to Y$ and $g: Y \to Z$ is homotopic to $g': Y \to Z$, then $g \circ f: X \to Z$ is homotopic to $g' \circ f': X \to Z$.

Exercise 7.2 Let X be a topological space. Show that X is path connected if and only if every two constant maps $c_1: X \to X$ and $c_2: X \to X$ are homotopic.

Exercise 7.3 Let X be a topological space, and let x_0, x_1 and x_2 be points in X. Show that if $\alpha: I \to X$ is a path in X from x_0 to x_1, $\beta: I \to X$ is a path in X from x_1 to x_2 and $\gamma = \alpha * \beta$, then $\widehat{\gamma} = \widehat{\beta} \circ \widehat{\alpha}$.

Exercise 7.4 Let S^2 be the 2-sphere considered to be a subspace of $\mathbb{R}^3$ with the standard topology. Show that $\pi_1(S^2 \setminus \{(0,0,1)\}, s)$ is the trivial group where $s = (0,0,-1)$.

Exercise 7.5 Let X be a Hausdorff space, and let A be a subspace of X. Show that if A is a retract of X, then A is a closed subset of X.

Exercise 7.6 Show that the relation of homotopy equivalence is an equivalence relation on any set of topological spaces.

A topological space X is said to be *contractible* if the identity map of X, $\mathrm{id}_X: X \to X$, is nullhomotopic.

Exercise 7.7 Show that a topological space X is contractible if and only if X has the homotopy type of a point, i.e., a topological space consisting of one point.

Exercise 7.8 Let X be a topological space and let S^1 be the unit circle considered as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Show that if $f: X \to S^1$ is a continuous map that is not surjective, then f is nullhomotopic.

8. The fundamental group of the circle

8.1 Covering spaces

The notion of a *covering space* is a helpful tool for determining fundamental groups. Specifically, we will compute the fundamental group of the circle by first establishing that the map $p\colon \mathbb{R} \to S^1$, given by

$$p(t) = (\cos(2\pi t), \sin(2\pi t)),$$

is a *covering map*.

Definition 8.1 (Covering spaces) Let B and E be topological spaces. We say that a surjective continuous map $p\colon E \to B$ is a *covering map* if for each point $b \in B$ there is a neighborhood U such that $p^{-1}(U)$ is a disjoint union of open subsets V_λ of E where $\lambda \in \Lambda$,

$$p^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} V_\lambda,$$

and $p|_{V_\lambda}\colon V_\lambda \to U$ is a homeomorphism for each $\lambda \in \Lambda$. We refer to E as a *covering space* of B.

A covering map $p\colon E \to B$ is sometimes also referred to as a *covering projection*. Note that B is often referred to as the *base space* of the covering map and that E is often referred to as the *total space* of the covering map. An open subset U of B satisfying the condition that

$$p^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} V_\lambda, \qquad \text{where } V_\lambda \text{ is an open subset of } E,$$

such that $p|_{V_\lambda}\colon V_\lambda \to U$ is a homeomorphism for each $\lambda \in \Lambda$ is said to be *evenly covered* by p. See Figure 8.1.

For each $b \in B$, we often refer to the preimage $p^{-1}(b)$ of $\{b\}$ under p as the *fiber* over b. Note also that the fiber over b is a discrete subspace of E; each V_λ is an open subset of E and intersects the fiber over b in a single point, and so, this point is open in the fiber over b. Finally, we note that a covering map is a special case of a *fiber bundle* with discrete fibers.

Example 8.2 Let B and E be topological spaces. If $p\colon E \to B$ is a homeomorphism, then p is also a covering map.

Example 8.3 Let X be a topological space, and let D be a discrete space. Then the projection map onto the first factor, $\mathrm{pr}_1\colon X \times D \to X$, where $X \times D$ is given the product topology, is a covering map. In particular, X is evenly covered by pr_1.

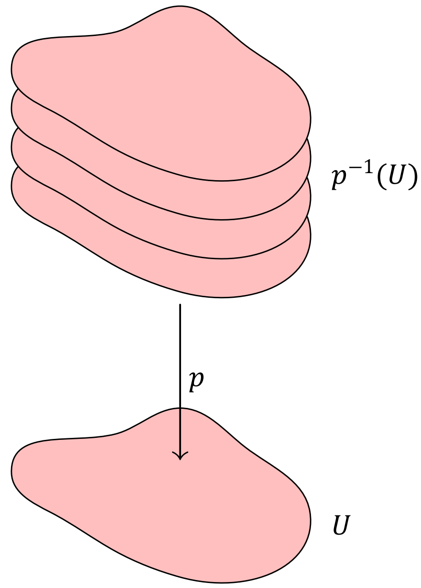


Figure 8.1: The open subset U of B is evenly covered by $p\colon E \to B$.

Definition 8.4 (Local homeomorphisms) Let X and Y be topological spaces. A continuous map $f\colon X \to Y$ is said to be a *local homeomorphism* if for each $x \in X$ there is a neighborhood U such that $f(U)$ is open in Y and $f|_U\colon U \to f(U)$ is a homeomorphism where U and $f(U)$ are both given the subspace topology from X and Y, respectively.

Example 8.5 Let X and Y be topological spaces, and let $f\colon X \to Y$ be a homeomorphism. Then f is also a local homeomorphism.

Local homeomorphisms preserve local topological properties such as local connectedness and local path connectedness. In other words, if $f\colon X \to Y$ is a local homeomorphism and, say, X is locally connected then so is Y.

A topological space X is *locally (path) connected at* $x \in X$ if for each neighborhood U of x there is a (path) connected neighborhood V of x such that $V \subseteq U$. We say that X is *locally (path) connected* if X is locally (path) connected at each $x \in X$. Note that a (path) connected space need not be locally (path) connected.

Clearly, a covering map is also a local homeomorphism. Thus if $p\colon E \to B$ is a covering map, then the total space and the base space are *locally* equivalent but they can differ *globally*.

The following theorem provides an important example of a covering space that we will use later on to compute the fundamental group of the circle.

Theorem 8.6 *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and consider the circle S^1 as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Then the map $p\colon \mathbb{R} \to S^1$ given by*

$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

is a covering map.

Proof. Clearly, p is continuous and surjective. Let $U = S^1 \setminus \{(-1, 0)\}$ and $V = S^1 \setminus \{(1, 0)\}$. Then both U and V are open in S^1, and their union is equal to S^1. We want to show that they are evenly covered by p.

The preimage of U under p is the disjoint union of subsets $V_\lambda = (\lambda - 1/2, \lambda + 1/2)$ of $\mathbb{R}$ for $\lambda \in \mathbb{Z}$;

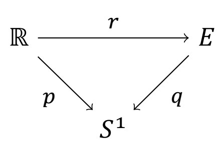
$$p^{-1}(U) = \bigsqcup_{\lambda \in \mathbb{Z}} V_\lambda = \bigsqcup_{\lambda \in \mathbb{Z}} \left(\lambda - \frac{1}{2}, \lambda + \frac{1}{2}\right).$$

Similarly, the preimage of V under p is the disjoint union of subsets $W_\lambda = (\lambda, \lambda + 1)$ of $\mathbb{R}$ for $\lambda \in \mathbb{Z}$;

$$p^{-1}(V) = \bigsqcup_{\lambda \in \mathbb{Z}} W_\lambda = \bigsqcup_{\lambda \in \mathbb{Z}} (\lambda, \lambda + 1).$$

Since p is continuous and open, cf. [5, Lemma 14.33], and V_λ and W_λ are open subsets of $\mathbb{R}$, it follows that $p|_{V_\lambda} \colon V_\lambda \to U$ and $p|_{W\lambda} \colon W_\lambda \to V$ are open maps for each $\lambda \in \mathbb{Z}$. Moreover, they are both also bijective for each $\lambda \in \mathbb{Z}$. Hence, by Theorem 5.21 they are both homeomorphisms. Thus $p \colon \mathbb{R} \to S^1$ is a covering map. □

The covering map p from Theorem 8.6 is illustrated in Figure 8.2. Theorem 8.6 also provides an example of a *universal covering space* since $\mathbb{R}$ is simply connected. It is universal in the following sense: for any covering map $q \colon E \to S^1$ where we assume that E is path connected, there is a covering map $r \colon \mathbb{R} \to E$ such that $p = q \circ r$.



> **Theorem 8.7** *Let B_1, B_2, E_1 and E_2 be topological spaces, and let $p_1 \colon E_1 \to B_1$ and $p_2 \colon E_2 \to B_2$ be covering maps. Then*
> $$p_1 \times p_2 \colon E_1 \times E_2 \to B_1 \times B_2$$
> *is a covering map where both $B_1 \times B_2$ and $E_1 \times E_2$ are given the product topology.*

Proof. Let $(b_1, b_2) \in B_1 \times B_2$ and let U_1 and U_2 be neighborhoods of b_1 and b_2, respectively, that are covered evenly by p_1 and p_2, respectively. We want to show that $U_1 \times U_2$ is evenly covered by $p_1 \times p_2$.

If

$$p_1^{-1}(U_1) = \bigsqcup_{\lambda \in \Lambda} V_\lambda \quad \text{and} \quad p_2^{-1}(U_2) = \bigsqcup_{\omega \in \Omega} W_\omega$$

then, clearly,

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2) = \bigsqcup_{\substack{\lambda \in \Lambda \\ \omega \in \Omega}} (V_\lambda \times W_\omega).$$

Furthermore, $p_1 \times p_2|_{V_\lambda \times W_\omega} \colon V_\lambda \times W_\omega \to U_1 \times U_2$ is a homeomorphism for each $\lambda \in \Lambda$ and each $\omega \in \Omega$. Hence, $U_1 \times U_2$ is a neighborhood of (b_1, b_2) that is evenly covered by $p_1 \times p_2$. Thus $p_1 \times p_2$ is a covering map. □

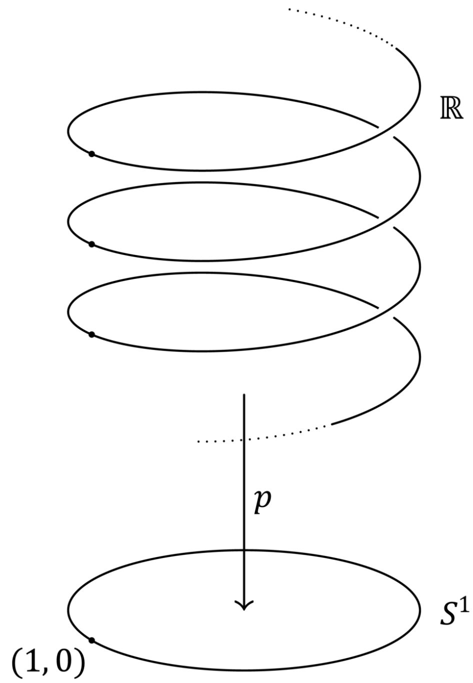


Figure 8.2: The covering map $p\colon \mathbb{R} \to S^1$ given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$.

Example 8.8 From Theorem 8.6 and Theorem 8.7, it follows that

$$p \times p\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$

where p is the covering map of Theorem 8.6, is a covering map. Hence, $\mathbb{R}^2$ is a (universal) covering space of the torus $T^2 = S^1 \times S^1$.

8.2 Computing the fundamental group of the circle

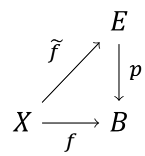
In this section we will see how we can use covering spaces as a tool to compute fundamental groups. In particular, we will compute the fundamental group of the circle. By Theorem 8.6, we know that the map $p\colon \mathbb{R} \to S^1$ given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$ is a covering map.

In order to compute $\pi_1(S^1, s_0)$ we will show that the *lifting correspondence*

$$\Phi\colon \pi_1(S^1, s_0) \to p^{-1}(s_0) = \mathbb{Z},$$

where $s_0 = (1, 0)$, is a bijection, and moreover, that it is an isomorphism of groups. This will involve certain *lifting* theorems.

Let B, E and X be topological spaces, and let $p\colon E \to B$ be a continuous map. If we are given a continuous map $f\colon X \to B$, we are often interested in whether or not there is a continuous map $\tilde{f}\colon X \to E$ such that $f = p \circ \tilde{f}$.



This is often referred to as a *lifting problem* for f.

Definition 8.9 **(Liftings)** Let B, E and X be topological spaces, and let $p\colon E \to B$ be a continuous map. A *lifting* of a continuous map $f\colon X \to B$ is a continuous map $\tilde{f}\colon X \to E$ such that $f = p \circ \tilde{f}$.

We will focus on the case where p is assumed to be a covering map. We will need a theorem that says that any path in the base space of a covering map can be lifted to a path in the total space. In order to prove the theorem we will need the following lemma.

Lemma 8.10 **(Lebesgue number lemma)** *Let (X, d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then there is a real number $\lambda > 0$ such that for every $x \in X$ there is a $U \in \mathcal{A}$ where $\mathrm{B}(x; \lambda) \subseteq U$.*

The number λ is called a *Lebesgue number* for the cover $\mathcal{A}$. The lemma says that for a given open cover of a compact metric space, there is a Lebesgue number such that every open ball with a radius less than the Lebesgue number must lie in some set in the open cover.

Proof. If $X \in \mathcal{A}$, then every real number $\lambda > 0$ will be a Lebesgue number for $\mathcal{A}$. So assume that X is not an element of $\mathcal{A}$.

By compactness, there is a finite subcollection, say, $\{U_1, U_2, \dots, U_n\}$ of $\mathcal{A}$ that covers X. For each $i \in \{1, 2, \dots, n\}$, let $V_i = X \setminus U_i$. Then each V_i is non-empty as $U_i \neq X$ for all $i \in \{1, 2, \dots, n\}$. Let $\mathbb{R}$ denote the set of real numbers equipped with the standard topology (and the standard Euclidean metric), and let $f\colon X \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{dist}_{V_i}(x)$$

where $\mathrm{dist}_{V_i}\colon X \to \mathbb{R}$ is given by

$$\mathrm{dist}_{V_i}(x) = \inf\{d(x, v) \mid v \in V_i\}.$$

We claim that dist_{V_i} is continuous for each $i \in \{1, 2, \dots, n\}$. Let x_1 and x_2 be two points in X. Then by M3, we have

$$\mathrm{dist}_{V_i}(x_1) \leqslant d(x_1, v) \leqslant d(x_1, x_2) + d(x_2, v),$$

for all $v \in V_i$, and so,

$$\mathrm{dist}_{V_i}(x_1) - d(x_1, x_2) \leqslant \mathrm{dist}_{V_i}(x_2).$$

Hence, $|\,\mathrm{dist}_{V_i}(x_1) - \mathrm{dist}_{V_i}(x_2)| \leqslant d(x_1, x_2)$. Thus dist_{V_i} is continuous for each $i \in \{1, 2, \dots, n\}$. Hence, f is also continuous.

We claim that $f(x) > 0$ for all $x \in X$. Choose an $x \in X$. Since $\{U_1, U_2, \dots, U_n\}$ covers X, there must be a $j \in \{1, 2, \dots, n\}$ such that $x \in U_j$. Since $U_j \in \mathcal{A}$, we know that U_j must be an open set. Thus there is an $\epsilon > 0$ such that $\mathrm{B}(x;\epsilon) \subseteq U_j$, and hence, $\mathrm{dist}_{V_j}(x) > \epsilon$. It follows that $f(x) > \epsilon/n > 0$. Thus $f(x) > 0$ for all $x \in X$ as claimed.

Since f is continuous with a compact domain and $f(x) > 0$ for all $x \in X$, it follows by Theorem 6.43 that f must have a minimum value λ where $\lambda > 0$.

We need to show that λ is a Lebesgue number for $\{U_1, U_2, \dots, U_n\}$, and hence, for $\mathcal{A}$. Choose an $x \in X$. Suppose that there is no $U_i \in \{U_1, U_2, \dots, U_n\}$ such that $\mathrm{B}(x;\lambda) \subseteq U_i$. Then $\mathrm{dist}_{V_i}(x) < \lambda$ for all $i \in \{1, 2, \dots, n\}$, and thus $f(x) < \lambda$. This contradicts the fact that λ is the minimum value of f over X. Hence, there must be a $U_i \in \mathcal{A}$ such that $\mathrm{B}(x;\lambda) \subseteq U_i$. Thus λ is a Lebesgue number as claimed. □

Example 8.11 Let $I = [0,1] \subseteq \mathbb{R}$ where $\mathbb{R}$ is given the standard Euclidean metric. Then $\lambda = 1/4$ is a Lebesgue number for the open cover $\mathcal{A} = \{U_1, U_2\}$ where $U_1 = (-3, 5/6)$ and $U_2 = (0,2)$: if $0 \leqslant x \leqslant 1/2$ then $\mathrm{B}(x;\lambda) = (x-\lambda, x+\lambda) = (x - 1/4, x + 1/4) \subseteq U_1$, and if $1/2 \leqslant x \leqslant 1$ then $\mathrm{B}(x;\lambda) \subseteq U_2$.

Theorem 8.12 **(Unique path lifting property)** *Let B and E be topological spaces, and let $p\colon E \to B$ be a covering map such that $p(e_0) = b_0$ where $e_0 \in E$. For any path $f\colon I \to B$ where $f(0) = b_0$ there is a unique path $\tilde{f}\colon I \to E$ lifting f such that $\tilde{f}(0) = e_0$.*

Proof. For each $s \in I$, let U_s be a neighborhood of $f(s)$ that is evenly covered by p. The collection $\mathcal{A} = \{f^{-1}(U_s) \mid s \in I\}$ is then an open cover of I. Since I is a compact metric space, it follows from Lemma 8.10 that there is a Lebesgue number $\lambda > 0$ for $\mathcal{A}$. Consider a subdivision of I consisting of points

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

such that $s_i - s_{i-1} < 2\lambda$ for $1 \leqslant i \leqslant n$. Then each interval $[s_{i-1}, s_i]$ lies in one of the preimages $f^{-1}(U_s)$. Thus $f([s_{i-1}, s_i]) \subseteq U_s$ for some $s \in I$.

We must show that there is a lift $\tilde{f}$ of f such that $\tilde{f}(0) = e_0$ and that this lift is unique. We prove the existence of $\tilde{f}$ first. Let U be a neighborhood of $f(s)$ for some $s \in I$ that is evenly covered by p, and let

$$p^{-1}(U) = \bigsqcup_{\omega \in \Omega} V_\omega.$$

Note that if $e_i \in p^{-1}(f(s_i))$ then $e_i \in V_{\omega'}$ for a unique $\omega' \in \Omega$. Since $p|_{V_{\omega'}}\colon V_{\omega'} \to U$ is a homeomorphism, it follows that the map $\tilde{g}_i\colon [s_{i-1}, s_i] \to E$ given by

$$\tilde{g}_i(s) = \left(\left(p|_{V_{\omega'}}\right)^{-1} \circ f|_{[s_{i-1}, s_i]}\right)(s)$$

is continuous, and moreover, $p \circ \tilde{g}_i = f|_{[s_{i-1}, s_i]}$ and $\tilde{g}_i(s_{i-1}) = e_{i-1}$. Thus there is a continuous map $\tilde{g}_1\colon [s_0, s_1] \to E$ such that $p \circ \tilde{g}_1 = f|_{[s_0, s_1]}$ and $\tilde{g}_1(s_0) = e_0$. Similarly, there is a continuous map $\tilde{g}_2\colon [s_1, s_2] \to E$ such that $p \circ \tilde{g}_2 = f|_{[s_1, s_2]}$ and $\tilde{g}_2(s_1) = \tilde{g}_1(s_1) = e_1$. Continuing this process, we get for each $i \in \{2, 3, \dots, n\}$ a continuous map $\tilde{g}_i\colon [s_{i-1}, s_i] \to E$ such that $p \circ \tilde{g}_i = f|_{[s_{i-1}, s_i]}$ and $\tilde{g}_i(s_{i-1}) = \tilde{g}_{i-1}(s_{i-1}) = e_{i-1}$. By Lemma 7.3, we can combine these maps into a continuous map

$$\tilde{f}\colon I \to E$$

given by $\tilde{f}(s) = \tilde{g}_i(s)$ for $s \in [s_{i-1}, s_i]$. Then $\tilde{f}$ is a lift of f such that $\tilde{f}(0) = e_0$.

We conclude the proof by showing the uniqueness of $\tilde{f}$. Assume that there are two continuous maps $\tilde{f}_1 : I \to E$ and $\tilde{f}_2 : I \to E$ such that $p \circ \tilde{f}_1 = p \circ \tilde{f}_2 = f$ and $\tilde{f}_1(0) = \tilde{f}_2(0) = e_0$. We want to show that $\tilde{f}_1 = \tilde{f}_2$. Let

$$A = \{s \in I \mid \tilde{f}_1(s) = \tilde{f}_2(s)\}$$

and

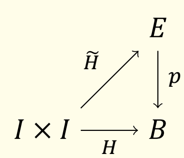
$$D = \{s \in I \mid \tilde{f}_1(s) \neq \tilde{f}_2(s)\}.$$

Then clearly, $I = A \cup D$ and $A \cap D = \emptyset$. If we can show that both A are D are open, then because I is connected we have $A = I$ and $D = \emptyset$ as $e_0 \in A$, and hence, $\tilde{f}_1 = \tilde{f}_2$.

Let $s \in A$, and let U be a neighborhood of $f(s)$ that is evenly covered by p. Assume that V_ω is the subset of $p^{-1}(U)$ that contains $\tilde{f}_1(s) = \tilde{f}_2(s)$. Let $\mathcal{U} = \tilde{f}_1^{-1}(V_\omega) \cap \tilde{f}_2^{-1}(V_\omega)$. Then $\mathcal{U}$ is a neighborhood of s and both $\tilde{f}_1$ and $\tilde{f}_2$ map all points in $\mathcal{U}$ to V_ω. Thus for all $t \in \mathcal{U}$, we have $\left(p \circ \tilde{f}_1\right)(t) = f(t) = \left(p \circ \tilde{f}_2\right)(t)$ implying that $\tilde{f}_1(t) = \tilde{f}_2(t)$ as $p|_{V_\omega} : V_\omega \to U$ is a homeomorphism. Hence, $t \in A$ and $\mathcal{U} \subseteq A$. Thus A is open.

Finally, we need to show that D is open. Choose an $s \in D$. Let U be a neighborhood of $f(s)$ that is evenly covered by p. Since $\tilde{f}_1(s) \neq \tilde{f}_2(s)$, there are unique indexes ω_1 and ω_2 in Ω such that $\tilde{f}_1(s) \in V_{\omega_1}$ and $\tilde{f}_2(s) \in V_{\omega_2}$. Then $\mathcal{U}' = \tilde{f}_1^{-1}(V_{\omega_1}) \cap \tilde{f}_2^{-1}(V_{\omega_2})$ is a neighborhood of s. Since V_{ω_1} and V_{ω_2} are disjoint, it follows that $\tilde{f}_1(t) \neq \tilde{f}_2(t)$ for all $t \in \mathcal{U}'$. Thus $\mathcal{U}' \subseteq D$. Hence, D is open. Thus $\tilde{f}_1 = \tilde{f}_2$. □

Theorem 8.13 **(Homotopy lifting property)** *Let B and E be topological spaces, and let $p : E \to B$ be a covering map such that $p(e_0) = b_0$ where $e_0 \in E$. If $H : I \times I \to B$ is a continuous map where $H(0,0) = b_0$, then there is a unique lifting $\widetilde{H} : I \times I \to E$ with $\widetilde{H}(0,0) = e_0$.*



Furthermore, if H is a path homotopy then $\widetilde{H}$ is a path homotopy.

Proof. We prove the statement about path homotopies. The proof of the first part of the theorem is similar to the proof of Theorem 8.12, where we make use of Lemma 8.10 to break $I \times I$ into smaller squares $[s_{i-1}, s_i] \times [t_{i-1}, t_i]$ that are mapped by H into open sets of B that are evenly covered by p. See [4, pp. 343–344] for details.

Assume that H is a path homotopy from the path $f_0 : I \to B$ to the path $f_1 : I \to B$ where $f_0(0) = f_1(0) = b_0$ and $f_0(1) = f_1(1) = b_1$. Then the map $H_0 : I \to B$ given by

$$H_0(t) = H(0,t)$$

is the constant path in B at b_0. Hence, the map $\widetilde{H}_0 : I \to E$ given by

$$\widetilde{H}_0(t) = \widetilde{H}(0,t)$$

is a lift of H_0 where $\widetilde{H}_0(0) = \widetilde{H}(0,0) = e_0$. Note that the constant path $c_{e_0} : I \to E$ at e_0 is also a lift of H_0 such that $c_{e_0}(0) = e_0$. Thus by uniqueness $\widetilde{H}_0 = c_{e_0}$. In other words, $\widetilde{H}(0,t) = e_0$ for all $t \in I$.

Similarly, let $H_1 : I \to B$ be the map given by

$$H_1(t) = H(1,t).$$

Then H_1 is the constant path c_{b_1} in B at b_1. Let $\widetilde{H}(1,0) = e_1$. Then the map $\widetilde{H}_1 : I \to E$ given by

$$\widetilde{H}_1(t) = \widetilde{H}(1,t)$$

is a lift of H_1 where $\widetilde{H}_1(0) = \widetilde{H}(1,0) = e_1$. As the constant path $c_{e_1} : I \to E$ at e_1 is also a lift of H_1 such that $c_{e_1}(0) = e_1$, it follows by uniqueness that $\widetilde{H}_1 = c_{e_1}$. In other words, $\widetilde{H}(1,t) = e_1$ for all $t \in I$. Thus $\widetilde{H}$ is a path homotopy from $\tilde{f}_0$ to $\tilde{f}_1$. □

In order to compute the fundamental group of the circle we will use a map referred to as the *lifting correspondence* that uses Theorem 8.13.

Definition 8.14 (Lifting correspondences) Let B and E be topological spaces, and let $p : E \to B$ be a covering map such that $p(e_0) = b_0$ where $e_0 \in E$. Furthermore, let $f : I \to B$ be a loop in B at b_0, and let $\tilde{f} : I \to E$ be the unique lift of f to a path in E such that $\tilde{f}(0) = e_0$. We say that the map

$$\Phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

given by $\Phi([f]) = \tilde{f}(1)$, is the *lifting correspondence* derived from p.

Let $f_0 : I \to B$ and $f_1 : I \to B$ be two loops in B at b_0 that are path homotopic. Then, by Theorem 8.13, the corresponding lifted paths $\tilde{f}_0 : I \to E$ and $\tilde{f}_1 : I \to E$ in E, where $\tilde{f}_0(0) = \tilde{f}_1(0) = e_0$, are also path homotopic. In particular, $\tilde{f}_0(1) = \tilde{f}_1(1)$. Thus Φ is a well-defined map. We also note that Φ depends on the choice of e_0 but we usually suppress this dependence in the notation, and hence, write Φ instead of Φ_{e_0}.

Theorem 8.15 *Let B be a topological space and let E be a path connected space. If $p : E \to B$ is a covering map such that $p(e_0) = b_0$ where $e_0 \in E$, then the lifting correspondence*

$$\Phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If we assume that E is simply connected, then Φ is a bijection.

Proof. Assume that E is path connected. We want to show that $\Phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. In other words, we want to show that for every point $e \in p^{-1}(b_0)$ there is a loop $f : I \to B$ in B based at b_0 such that $\Phi([f]) = e$. Since E is path connected there is a path $g : I \to E$ in E from e_0 to e. Let $f = p \circ g : I \to B$. Then $f(0) = p(g(0)) = p(e_0) = b_0$ and $f(1) = p(g(1)) = p(e) = b_0$. Hence, f is a loop in B based at b_0 and, by Theorem 8.12, g is the unique lift of f to a path in E starting at e_0, i.e., $g = \tilde{f}$. Thus $\Phi([f]) = \tilde{f}(1) = e$. Hence, Φ is surjective.

Now assume that E is simply connected. We want to show that in this case Φ is also injective. In other words, if $f : I \to B$ and $g : I \to B$ are two loops in B based at b_0 where $\Phi([f]) = \Phi([g]) = e$, then we want to show that $[f] = [g]$. Let $\tilde{f}$ and $\tilde{g}$ be the unique lifts of f and g, respectively, such that $\tilde{f}(0) = \tilde{g}(0) = e_0$. Since we have assumed that $\Phi([f]) = \Phi([g]) = e$, it follows that $\tilde{f}(1) = \tilde{g}(1) = e$. By taking the product of $\tilde{f}$ and the reverse of $\tilde{g}$, $\tilde{f} * \overline{\tilde{g}}$, we get a loop in E based at e_0. Since E is simply connected there must be a path homotopy H from $\tilde{f} * \overline{\tilde{g}}$ to the constant path c_{e_0}

in E at e_0. It follows that the composition $p \circ H$ is a path homotopy from $f * \overline{g}$ to the constant path c_{b_0} in B at b_0. Thus $[f] * [g]^{-1} = e$ where e denotes the identity element in the group $\pi_1(B, b_0)$, and hence, $[f] = [g]$. Hence, Φ is injective and thus bijective. □

We now have all the tools we need to compute the fundamental group of the circle.

Theorem 8.16 *Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle considered as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Then for any $s_0 \in S^1$, the lifiting correspondence*

$$\Phi \colon \pi_1(S^1, s_0) \to \mathbb{Z}$$

is an isomorphism where $\mathbb{Z}$ is the additive group of integers.

Proof. Since S^1 is path connected, it follows by Theorem 7.19 that we only need to compute $\pi_1(S^1, s_0)$ for one base point (and hence, all) $s_0 \in S^1$. By Theorem 8.6, we know that the map $p \colon \mathbb{R} \to S^1$ given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$ is a covering map. Furthermore, since $\mathbb{R}$ is simply connected, it follows by Theorem 8.15 that the lifting correspondence

$$\Phi \colon \pi_1(S^1, s_0) \to p^{-1}(s_0)$$

is a bijection. Let $e_0 = 0$ such that $p(e_0) = p(0) = (1, 0) = s_0$. Clearly, $p^{-1}(s_0) = \mathbb{Z}$.

We want to show that Φ is a homomorphism. In other words, if $[f], [g] \in \pi_1(S^1, s_0)$ we want to show that $\Phi([f] * [g]) = \Phi([f]) + \Phi([g])$. Assume that $f \colon I \to S^1$ and $g \colon I \to S^1$ are two loops in S^1 based at s_0. By Theorem 8.12 there are unique lifts $\tilde{f}$ and $\tilde{g}$ of f and g, respectively, such that $\tilde{f}(0) = \tilde{g}(0) = 0$. Let $\tilde{f}(1) = x$ and $\tilde{g}(1) = y$ such that, by definition of Φ, we have $\Phi([f]) = x$ and $\Phi([g]) = y$. We will show that $\Phi([f * g]) = x + y$.

To compute $\Phi([f * g])$ we must lift $f * g$ to a path $\widetilde{f * g}$ in $\mathbb{R}$ such that $(\widetilde{f * g})(0) = 0$. Let $\tilde{h} \colon I \to \mathbb{R}$ be a path in $\mathbb{R}$ given by

$$\tilde{h}(s) = \tilde{f}(1) + \tilde{g}(s) = x + \tilde{g}(s).$$

Note that $\tilde{h}$ is a lift of g starting at x. Let $\tilde{f} * \tilde{h} \colon I \to \mathbb{R}$ be a path in $\mathbb{R}$ given by

$$(\tilde{f} * \tilde{h})(s) = \begin{cases} \tilde{f}(2s) & 0 \leqslant s \leqslant 1/2 \\ \tilde{h}(2s - 1) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Since $p(\tilde{f} * \tilde{h}) = f * g$, it follows that $\tilde{f} * \tilde{h}$ is the unique lift of $f * g$ such that $(\tilde{f} * \tilde{h})(0) = 0$. In other words, $\widetilde{f * g} = \tilde{f} * \tilde{h}$. Finally, since $\Phi([f] * [g]) = \Phi([f * g]) = (\widetilde{f * g})(1) = \tilde{h}(1) = x + \tilde{g}(1) = x + y$, it follows that $\Phi([f] * [g]) = \Phi([f]) + \Phi([g])$. Thus Φ is a homomorphism. □

As an application of Theorem 8.16, we can prove the Brouwer fixed point theorem in dimension 2, cf. Theorem 1.1.

Theorem 8.17 **(Brouwer fixed point theorem in dimension** 2**)** *Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ be the disk considered as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Then every continuous map $f \colon D^2 \to D^2$ has a fixed point, i.e., there is an $x \in D^2$ such that $f(x) = x$.*

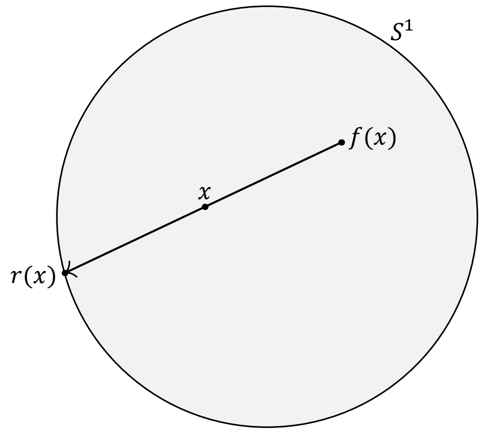


Figure 8.3: The map $r\colon D^2 \to S^1$.

Proof. Assume that f has no fixed points. Then there is a retraction $r\colon D^2 \to S^1$ given by having $r(x)$ be the point on the ray that originates at $f(x)$ and passes through x that lies on the circle. See Figure 8.3.

By Lemma 7.30, the homomorphism $i_*\colon \pi_1(S^1, s_0) \to \pi_1(D^2, s_0)$ induced by the inclusion map $i\colon S^1 \to D^2$ must be a monomorphism. Since D^2 is simply connected (it has the homotopy type of a point), it follows by Theorem 8.16 that the homomorphism i_* cannot be a monomorphism. Hence, there is no retraction $r\colon D^2 \to S^1$. □

8.3 The fundamental theorem of algebra

The fundamental theorem of algebra, cf. Theorem 1.2, can be proved in many different ways. In fact, there is a book, [2], that is entirely dedicated into proving the fundamental theorem of algebra and that provides a total of twelve different ways (including a modern version of Gauss's original proof from 1799) of proving the theorem! These proofs use techniques from abstract algebra, complex analysis and topology.

We will now present one proof (which is not identical with any of the ones given in [2]) that uses what we know about the fundamental group of the circle. The proof we give here borrows freely from the one found in [4, §56] and it will consist of four steps. Throughout this section we will think of S^1 as the unit circle in the complex numbers $\mathbb{C}$. In other words, S^1 is the set of points in $\mathbb{C}$ with modulus 1, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We consider S^1 as a subspace of $\mathbb{C}$ where $\mathbb{C} \cong \mathbb{R}^2$ is given the standard topology.

Step 1: Let $f\colon S^1 \to S^1$ be the map given by $f(z) = z^n$ where $n \in \mathbb{Z}_+$. Then the induced homomorphism

$$f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$$

is a monomorphism (note that $1 \in \mathbb{C}$ corresponds to $s_0 = (1, 0) \in \mathbb{R}^2$).

Step 2: Let $g\colon S^1 \to \mathbb{C} \setminus \{0\}$ be the map given by $g(z) = z^n$. Then g is not nullhomotopic, i.e., g is not homotopic to a constant map.

Step 3: *Special case.* Let

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

be a polynomial equation with coefficients in $\mathbb{C}$ where we assume that

$$|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1.$$

Then the polynomial equation has a root residing in $D^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$.

Step 4: *General case.* Let

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

be a polynomial equation with coefficients in $\mathbb{C}$. Then the polynomial equation has at least one complex root.

We will need the following lemma to aid us with the second and third step in the proof of Theorem 1.2.

Lemma 8.18 *Let X be a topological space, and let $h\colon S^1 \to X$ be a continuous map such that $h(1) = x_0$. Then h is nullhomotopic if and only if the induced homomorphism $h_*\colon \pi_1(S^1, 1) \to \pi_1(X, x_0)$ is the trivial homomorphism.*

Proof. Assume that h is nullhomotopic. Specifically, let $h \simeq c$ where c is the constant map $c\colon S^1 \to X$ given by $c(z) = x$. Then by Lemma 7.37 there is a path $\alpha\colon I \to X$ from x_0 to x such that $c_* = \widehat{\alpha} \circ h_*$. Since $\widehat{\alpha}$ is an isomorphism (cf. the proof of Theorem 7.19), it follows that since c_* is the trivial homomorphism, then so is h_*.

Now assume that h_* is the trivial homomorphism. Let $\gamma\colon I \to S^1$ be the loop in S^1 based at 1 given by

$$\gamma(s) = e^{2\pi i s}.$$

As the group of integers (with addition as its binary operation) is cyclic, i.e., $\mathbb{Z} = \langle 1 \rangle$, cf. Example A.10, and $\pi_1(S^1, 1) \cong \mathbb{Z}$, it follows that $[\gamma]$ is a generator for $\pi_1(S^1, 1)$. Since h_* is the trivial homomorphism, it follows that $h_*([\gamma]) = [h \circ \gamma] = e \in \pi_1(X, x_0)$ where e is the identity element. Thus there is a path homotopy $F\colon I \times I \to X$ from $h \circ \gamma$ to c_{x_0} where c_{x_0} is the constant path in X at x_0. In particular,

$$F(0, t) = F(1, t) = x_0$$

for all $t \in I$. We want to show that there is a homotopy $H\colon S^1 \times I \to X$ from h to the constant map $c\colon S^1 \to X$ given by $c(z) = x_0$. Since $\gamma\colon I \to S^1$ is a surjective continuous map whose domain is compact and whose codomain is Hausdorff, it follows that it is a quotient map. Thus $\gamma \times \mathrm{id}\colon I \times I \to S^1 \times I$ is also a quotient map where $\mathrm{id}\colon I \to I$ is the identity map. Then F induces a continuous map $H\colon S^1 \times I \to X$, where $F = H \circ (\gamma \times \mathrm{id})$, that is a homotopy from h to c. Hence, h is nullhomotopic. ☐

We now have everything we need to prove Theorem 1.2.

Proof of Theorem 1.2. Step 1: Let $f\colon S^1 \to S^1$ be the map given by $f(z) = z^n$, and let $\gamma\colon I \to S^1$ be the loop in S^1 based at 1 given by

$$\gamma(s) = e^{2\pi i s}.$$

Then the loop $f \circ \gamma\colon I \to S^1$, given by

$$(f \circ \gamma)(s) = f(\gamma(s)) = \left(e^{2\pi i s}\right)^n = e^{2\pi i n s},$$

lifts to the path $\theta\colon I \to \mathbb{R}$, given by $\theta(s) = ns$, as

$$(p \circ \theta)(s) = p(\theta(s)) = p(ns) = e^{2\pi i n s} = (f \circ \gamma)(s)$$

where $p\colon \mathbb{R} \to S^1$ is the map given by $p(t) = e^{2\pi i t}$ which we know to be a covering map, cf. Theorem 8.6.

Let $\Phi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ be the lifting correspondence derived p, cf. Theorem 8.16. Then

$$\Phi([f \circ \gamma]) = \theta(1) = n$$

and as γ lifts to the path $\tilde{\gamma}\colon I \to \mathbb{R}$ given by $\tilde{\gamma}(s) = s$, we have

$$\Phi([\gamma]) = \tilde{\gamma}(1) = 1.$$

By definition of the induced homomorphism $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$, we have

$$f_*([\gamma]) = [f \circ \gamma].$$

Since $\Phi([f \circ \gamma]) = n$ this means that if $[\alpha], [\beta] \in \pi_1(S^1, 1)$ where $f_*([\alpha]) = [\beta]$, we have $\Phi([\beta]) = n\Phi([\alpha])$ (here we use the fact that $\pi_1(S^1, 1)$ is cyclic and generated by $[\gamma]$), and so, $\Phi(f_*([\alpha])) = n\Phi([\alpha])$. In this sense, we can speak of f_* as "multiplication by n."

We want to show that f_* is injective. Let $[\alpha_1], [\alpha_2] \in \pi_1(S^1, 1)$ such that $[\alpha_1] \neq [\alpha_2]$. Since Φ is bijective, and hence, injective, it follows that $\Phi([\alpha_1]) \neq \Phi([\alpha_2])$. Furthermore, since $n \in \mathbb{Z}_+$ we have

$$\Phi(f_*([\alpha_1])) = n\Phi([\alpha_1]) \neq n\Phi([\alpha_2]) = \Phi(f_*([\alpha_2])).$$

Thus $[\alpha_1] \neq [\alpha_2]$ implies that $\Phi(f_*([\alpha_1])) \neq \Phi(f_*([\alpha_2]))$. In other words, f_* is injective.

Step 2: Let $g\colon S^1 \to \mathbb{C} \setminus \{0\}$ be the map given by $g(z) = z^n$. Then if $j\colon S^1 \to \mathbb{C} \setminus \{0\}$ is the inclusion map, we have $g = j \circ f$. By Example 7.29 and Lemma 7.30, the induced homomorphism $j_*\colon \pi_1(S^1, 1) \to \pi_1(\mathbb{C} \setminus \{0\}, 1)$ is a monomorphism, and by Theorem 7.24 so is g_* as

$$g_* = (j \circ f)_* = j_* \circ f_*\colon \pi_1(S^1, 1) \to \pi_1(\mathbb{C} \setminus \{0\}, 1)$$

i.e., g_* is the composition of two monomorphisms. As $\pi_1(S^1, 1) \cong \mathbb{Z}$, the monomorphism g_* cannot be the trivial homomorphism. Thus by Lemma 8.18 g is not nullhomotopic.

Step 3: Consider the polynomial equation given by

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

where we assume that the coefficients $a_0, a_1, \ldots a_{n-1}$ are complex numbers and that

$$|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1.$$

Assume that the polynomial equation has no root in D^2. We will show that this leads to a contradiction to the fact that the map g defined in the previous step is not nullhomotopic.

By assumption that the polynomial equation has no root in D^2, we may define a map $k\colon D^2 \to \mathbb{C} \setminus \{0\}$ given by

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0.$$

Then, clearly, k is continuous. Let $h = k|_{S^1}$. Since h can be extended to k, i.e., we have $h = k \circ i$ where $i\colon S^1 \to D^2$ is the inclusion map, it follows by Lemma 8.18 that h is nullhomotopic as $h_* = (k \circ i)_* = k_* \circ i_*$ must be the trivial homomorphism.

We now show that there is a homotopy from g to h, and so, g must be nullhomotopic as h is nullhomotopic. This contradicts our findings from the previous step, and hence, the polynomial equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

where we assume that the coefficients $a_0, a_1, \ldots, a_{n-1}$ are complex numbers and that

$$|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1$$

must have a root in D^2.

Let $F\colon S^1 \times I \to \mathbb{C} \setminus \{0\}$ be the map given by

$$F(z,t) = z^n + t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0).$$

Then F is clearly continuous and $F(z,0) = z^n = g(z)$ and $F(z,1) = z_n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = h(z)$ for all $z \in S^1$. Note that since $g(z) \neq 0$ for all $z \in S^1$ and $h(z) \neq 0$ for all $z \in S^1$. Since

$$\begin{aligned}
|F(z,t)| &= |z^n + t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0)| \\
&\geqslant |z^n| - |t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0)| \\
&\geqslant |z^n| - t(|a_{n-1}z^{n-1}| + |a_{n-2}z^{n-2}| + \cdots + |a_1 z| + |a_0|) \\
&= 1 - t(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|)
\end{aligned}$$

where the two inequalities follow from the triangle inequality and the following equality follows from the fact that $|z| = 1$ as $z \in S^1$, it follows from the assumption that

$$|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| < 1$$

and that $0 < t < 1$ that $|F(z,t)| > 0$ for all $z \in S^1$ and all $0 < t < 1$. Hence, F is a homotopy from g to h.

Step 4: For the final step we need to show that we can extend from the special case in the previous step to the general case. Let

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

be a polynomial equation with complex coefficients.

In order to use the previous step we need to “scale” the coefficients such that we have a similar assumption of the sum of the modulus of the coefficients as in the previous step. To do this, choose a (real) number $s > 0$ large enough so that

$$\left|\frac{a_{n-1}}{s}\right| + \left|\frac{a_{n-2}}{s^2}\right| + \cdots + \left|\frac{a_1}{s^{n-1}}\right| + \left|\frac{a_0}{s^n}\right| < 1.$$

Let $z = sw$ such that the polynomial equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

now becomes

$$s^n w^n + a_{n-1}s^{n-1}w^{n-1} + \cdots + a_1 s w + a_0 = 0.$$

Dividing by s^n we get the polynomial equation

$$w^n + \frac{a_{n-1}}{s}w^{n-1} + \cdots + \frac{a_1}{s^{n-1}}w + \frac{a_0}{s^n} = 0$$

which, by the previous step, must have a root, say, w_0 in D^2. Hence, $z_0 = sw_0$ is a root for the polynomial equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

This completes the proof. □

The property that every complex non-constant polynomial has a complex root is often referred to as the (field of) complex numbers being *algebraically closed*. The field of real numbers is, however, *not* algebraically closed, as, e.g., the polynomial equation $x^2 + 1 = 0$ has no root in $\mathbb{R}$.

8.4 Exercises

Exercise 8.1 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and consider S^1 as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Show that the product space $\mathbb{R} \times S^1$ is a covering space of the torus $T^2 = S^1 \times S^1$.

Exercise 8.2 Let B be a simply connected space, and let E be a path connected space. Show that if $p: E \to B$ is a covering map, then p is a homeomorphism.

Exercise 8.3 Let B and E be two topological spaces, and let $e_0 \in E$. Show that if $p: E \to B$ is a covering map such that $p(e_0) = b_0$, then the induced homomorphism $p_*: \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

Exercise 8.4 Let B be a Hausdorff space, and let E be a topological space. Show that if $p: E \to B$ is a covering map, then E must be Hausdorff.

Exercise 8.5 Let S^1 be the unit circle considered as a subspace of the complex numbers $\mathbb{C}$ where $\mathbb{C} \cong \mathbb{R}^2$ is given the standard topology. Show that for every integer n greater than 1 the map $p: S^1 \to S^1$ given by

$$p(z) = z^n$$

is a covering map.

Exercise 8.6 Let $\mathbb{R}$ denote the set of real numbers equipped with the standard topology, and let $\mathbb{Z}$ denote the set of integers. Consider $E = \{(x, y) \in \mathbb{R}^2 \mid x - y \in \mathbb{Z}\}$ as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology. Show that the map $p: E \to \mathbb{R}$ given by $p(x, y) = x$ is a covering map.

Exercise 8.7 Let n be a positive integer that is greater than or equal to 2. Show that for any $x_0 \in \mathbb{R}\mathrm{P}^n$

$$\pi_1(\mathbb{R}\mathrm{P}^n, x_0) \cong \mathbb{Z}/2$$

where $\mathbb{R}\mathrm{P}^n$ is real projective n-space. (You may assume as a known fact that the n-sphere S^n (considered as a subspace of $\mathbb{R}^{n+1}$ where $\mathbb{R}^{n+1}$ is given the standard topology) is simply connected for $n \geqslant 2$.)

Exercise 8.8 Let $A_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 \mid a \leqslant \sqrt{x_1^2 + x_2^2} \leqslant b\}$ be considered as a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ is given the standard topology and $a, b \in \mathbb{R}$ with $0 < a < 1 < b$. Compute $\pi_1(A_{a,b}, x_0)$ for any $x_0 \in A_{a,b}$.

A. Elementary algebra

A.1 Groups

Groups are sets with a binary operation satisfying certain axioms, and frequently appear in relation to symmetry.

Definition A.1 (Binary operations) Let S be a set. A *binary operation* on S is a map $*\colon S\times S \to S$. We write $a * b$ instead of $*(a,b)$ for $a, b \in S$.

The definition says that the set S is *closed* under the binary operation $*$ in the sense that if a and b are elements in S, then so is $a * b$.

Example A.2 Let $\mathbb{Z}$ be the set of integers. Addition $(+)$ and multiplication $(\cdot)$ are two binary operations on $\mathbb{Z}$.

Definition A.3 (Groups) A group is a non-empty set G together with a binary operation $*$ such that the following properties hold.

G1 For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

G2 There is an element $e \in G$ such that $a * e = a = e * a$ for all $a \in G$.

G3 For each $a \in G$ there is an element $a' \in G$ such that $a * a' = e = a' * a$.

We say that a group is *abelian* if $*$ is also commutative: $a * b = b * a$ for all $a, b \in G$.

A group is strictly speaking an ordered pair $(G, *)$. We often omit specific mention of $*$ if no confusion will arise. The first axiom, G1, says that $*$ is an associative operation. The second axiom, G2, says that there is a *neutral* or *identity element* for $*$. The third axiom, G3, says that for each element $a \in G$ there is an *inverse element* $a' \in G$ such that a' is both a *left* and *right inverse element*, i.e., $a' * a = e$ and $a * a' = e$, respectively. The *order* of a group G is the number of elements in G, written $|G|$.

We say that a group is written *multiplicatively* if we write ab instead of $a*b$, 1 (sometimes) instead of e and a^{-1} instead of a', and we say that a group is written *additively* if we write $a+b$ instead of $a*b$, 0 (sometimes) instead of e and $-a$ instead of a'.

Example A.4 The set of real numbers, $\mathbb{R}$, is an abelian group under addition: G1, G2 and G3 are all satisfied with $e = 0$ and the inverse element of $a \in \mathbb{R}$ being $-a$. Furthermore, it is an abelian group as $a + b = b + a$ for all $a, b \in \mathbb{R}$.

Note that $\mathbb{R}$ is *not* a group under multiplication: while G1 and G2 are satisfied (with $e = 1$), G3 fails. There is no real number we can multiply with 0 to get 1.

Example A.5 For $n \in \mathbb{Z}_+$, let

$$\mathbb{Z}/n = \{0, 1, 2, \dots, n-1\}.$$

Then $\mathbb{Z}/n$ is an abelian group with $+_n$ (addition modulo n) as its binary operation, i.e., $a +_n b = a + b \mod n$ for $a, b \in \mathbb{Z}/n$. Note that some authors write $\mathbb{Z}_n$, $Z/(n)$ or $\mathbb{Z}/n\mathbb{Z}$ for $\mathbb{Z}/n$.

Example A.6 Let $\mathrm{GL}_n(\mathbb{R})$ denote the set of invertible $n \times n$-matrices with entries in $\mathbb{R}$. Then $\mathrm{GL}_n(\mathbb{R})$ is a group with matrix multiplication as its binary operation. Note that it is *not* abelian, since, e.g.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition A.7 (Subgroups) Let G be a group with binary operation $*$. We say that a subset H of G is a *subgroup*, written $H \leqslant G$, if $*$ restricted to $H \times H$ is a binary operation on H such that H with $*|_{H \times H}$ is a group. We say that a subgroup H of G is a *proper subgroup* of G, written $H < G$, if H is a proper subset of G.

The set consisting of just the identity element is always a subgroup of a group. It is often called the *trivial subgroup*.

Example A.8 Let $G = \mathbb{Z}$ with addition as its binary operation. For $m \in \mathbb{Z}_+ \setminus \{1\}$, let

$$H = m\mathbb{Z} = \{mn \mid n \in \mathbb{Z}\} = \{\dots, -2m, -m, 0, m, 2m, \dots\}.$$

Then H is a proper subgroup of G.

Theorem A.9 *Let G be a group that is written multiplicatively, and let $x \in G$. If H is a subgroup of G containing x, then*

$$\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$$

is a subgroup of H.

We say that $\langle x \rangle$ is the *cyclic subgroup of G generated by x*. Note that $\langle x \rangle$ is the smallest subgroup of G containing x. If there is an $x' \in G$ such that $G = \langle x' \rangle$, we say that G is *cyclic*, and that x' is a *generator* for G. Note that cyclic groups may have more than one generators and that all cyclic groups are abelian. If the group is written additively, we interpret $\langle x \rangle$ as the set $\{nx \mid n \in \mathbb{Z}\}$.

Proof. Since H is a subgroup of G, it follows that $x^{-1} \in H$. Thus $xx^{-1} = x^{-1}x = x^0 = 1 \in H$. Furthermore, $x^n \in H$ for all $n \in \mathbb{Z}$ where $x^{-m} = (x^{-1})^m$. Thus $\langle x \rangle$ is a subgroup of H. □

Example A.10 The set of integers, $\mathbb{Z}$, is an infinite cyclic group with addition as its binary operation generated by both -1 and 1, i.e., $\langle -1 \rangle = \langle 1 \rangle = \mathbb{Z}$.

Definition A.11 **(Left and right cosets)** Let G be a group, and let H be a subgroup of G. For $g \in G$, we say that the set

$$g * H = \{g * h \mid h \in H\}$$

is the *left coset* of H containing g, and that the set

$$H * g = \{h * g \mid h \in H\}$$

is the *right coset* of H containing g.

Clearly, if G is abelian, the left and right cosets are equal. However, the left and right cosets may coincide even if G is not abelian.

Definition A.12 **(Normal subgroups)** Let G be a group. We say that a subgroup H of G is *normal* if the left and right cosets coincide, i.e., $g * H = H * g$ for all $g \in G$.

An equivalent definition of a normal subgroup is to say that H is a normal subgroup of G if $g * h * g^{-1} \in H$ for each $g \in G$ and each $h \in H$.

Theorem A.13 *Let G be a group written multiplicatively, and let H be a normal subgroup of G. Then the set of cosets of H, denoted G/H, is a group with binary operation given by*

$$(g_1H)(g_2H) = (g_1g_2)H$$

for $g_1, g_2 \in G$.

We refer to the group G/H as the *factor group* or *quotient group* of G by H. If G is a group, and H is a subgroup of G the relation $\sim_L$ given by $a \sim_L b$ if and only if $a^{-1}b \in H$ for $a, b \in G$ is an equivalence relation on G. It can be shown that for a fixed $g \in G$, the set $\{x \in G \mid g \sim_L x\}$ is equal to the left coset of H containing g, i.e., gH. It is a fact that if $g' \in gH$, then $g'H = gH$.

Proof. We first show that the binary operation defined on G/H is well-defined. Let $g_1' \in g_1H$, and let $g_2' \in g_2H$. Then $g_1H = g_1'H$ and $g_2H = g_2'H$. We must show that $(g_1H)(g_2H) = (g_1'H)(g_2'H)$. Choose an element $(g_1g_2)h \in (g_1g_2)H$. Since $g_1 \in g_1H = g_1'H$ and $g_2 \in g_2H = g_2'H$, there are elements h_1 and h_2 in H such that $g_1 = g_1'h_1$ and $g_2 = g_2'h_2$. Thus

$$\begin{aligned}
(g_1g_2)h &= (g_1'h_1)(g_2'h_2)h \\
&= g_1'h_1g_2'h_2h \\
&= g_1'(g_2'(g_2')^{-1})h_1g_2'h_2h \\
&= g_1'g_2'((g_2')^{-1}h_1g_2')h_2h \\
&= g_1'g_2'h_1'h_2h \in (g_1'g_2')H
\end{aligned}$$

where the last equality follows from the fact H is a normal subgroup of G; there is an $h_1' \in H$ such that $(g_2')^{-1}h_1g_2 = h_1'$. Hence, $(g_1g_2)H \subseteq (g_1'g_2')H$. A similar argument shows that $(g_1'g_2')H \subseteq (g_1g_2)H$. Hence, $(g_1g_2)H = (g_1'g_2')H$. Thus

$$(g_1H)(g_2H) = (g_1g_2)H = (g_1'g_2')H = (g_1'H)(g_2'H).$$

We now show that G/H is a group. Let g_1H, g_2H and g_2H be elements in G/H. We must show that $[(g_1H)(g_2H)](g_3H) = (g_1H)[(g_2H)(g_3H)]$. By definition, we have

$$\begin{aligned}[(g_1H)(g_2H)](g_3H) &= (g_1g_2H)(g_3H)\\ &= (g_1g_2)g_3H\\ &= g_1(g_2g_3)H\\ &= (g_1H)(g_2g_3H)\\ &= (g_1H)[(g_2H)(g_3H)].\end{aligned}$$

Hence, G1 is satisfied. If e is the identity element in G, then

$$(gH)(eH) = (ge)H = gH = (eg)H = (eH)(gH)$$

for all $g \in G$. Hence, G2 is satisfied with $eH = H$ as the identity element in G/H. Finally, for each $g \in G$, we have

$$(gH)(g^{-1}H) = (gg^{-1}H) = eH = (g^{-1}g)H = (g^{-1}H)(gH).$$

Hence, G3 is satisfied with $(gH)^{-1} = g^{-1}H$. Thus G/H is a group. □

The following theorem describes how we may construct a new group from two existing groups.

Theorem A.14 *Let G_1 and G_2 be groups with binary operations $*_1$ and $*_2$, respectively. Then $G_1 \times G_2$ is a group with binary operation given by*

$$(g_1, g_2) * (g_1', g_2') = (g_1 *_1 g_1', g_2 *_2 g_2')$$

for $g_1, g_1' \in G_1$ and $g_2, g_2' \in G_2$.

We say that that the group $G_1 \times G_2$ is the *direct product* of G_1 and G_2.

Proof. Since G_1 and G_2 are groups with binary operations $*_1$ and $*_2$, respectively, it follows that

$$(g_1, g_2) * (g_1', g_2') = (g_1 *_1 g_1', g_2 *_2 g_2') \in G_1 \times G_2$$

for $(g_1, g_2), (g_1', g_2') \in G_1 \times G_2$.

Let $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2) be elements in $G_1 \times G_2$. Then

$$\begin{aligned}[(a_1, a_2) * (b_1, b_2)] * (c_1, c_2) &= (a_1 *_1 b_1, a_2 *_2 b_2) * (c_1, c_2)\\ &= ((a_1 *_1 b_1) *_1 c_1, (a_2 *_2 b_2) *_2 c_2)\\ &= (a_1 *_1 (b_1 *_1 c_1), a_2 *_2 (b_2 *_2 c_2))\\ &= (a_1, a_2) * (b_1 *_1 c_1, b_2 *_2 c_2)\\ &= (a_1, a_2) * [(b_1, b_2) * (c_1, c_2)].\end{aligned}$$

Thus G1 is satisfied.

If e_1 and e_2 are the identity elements in G_1 and G_2, respectively, then

$$\begin{aligned}(g_1, g_2) * (e_1, e_2) &= (g_1 *_1 e_1, g_2 *_2 e_2) \\ &= (g_1, g_2)\end{aligned}$$

and

$$\begin{aligned}(e_1, e_2) * (g_1, g_2) &= (e_1 *_1 g_1, e_2 *_2 g_2) \\ &= (g_1, g_2)\end{aligned}$$

for all $(g_1, g_2) \in G_1 \times G_2$. Hence, G2 is satisfied.

Finally, if g_1^{-1} and g_2^{-1} denotes the inverse elements of elements $g_1 \in G_1$ and $g_2 \in G_2$, respectively, then

$$\begin{aligned}(g_1, g_2) * (g_1^{-1}, g_2^{-1}) &= \left(g_1 *_1 g_1^{-1}, g_2 *_2 g_2^{-1}\right) \\ &= (e_1, e_2)\end{aligned}$$

and

$$\begin{aligned}(g_1^{-1}, g_2^{-1}) * (g_1, g_2) &= \left(g_1^{-1} *_1 g_1, g_2^{-1} *_2 g_2\right) \\ &= (e_1, e_2).\end{aligned}$$

Thus G3 holds. Hence, $G_1 \times G_2$ with $*$ as its binary operation is a group. □

We can extend the theorem to hold for a finite collection of groups: if $G_1, G_2, \dots, G_n$ are groups, then

$$\prod_{i=1}^{n} G_i = G_1 \times G_2 \times \cdots \times G_n$$

is a group with binary operation as defined above suitably expanded. Note also that the direct product is abelian if and only if each of the factors are abelian.

A.2 Homomorphisms

Homomorphisms are structure preserving maps from one group to another.

Definition A.15 (Homomorphisms) Let G_1 and G_2 be groups with binary operations $*_1$ and $*_2$, respectively. Then a map $\varphi\colon G_1 \to G_2$ is a *homomorphism* if

$$\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$$

for all $x, y \in G_1$.

Note that if e_1 and e_2 are the identity elements in G_1 and G_2, respectively, then for any homomorphism $\varphi\colon G_1 \to G_2$, we have $\varphi(e_1) = e_2$. If x^{-1} denotes the inverse element of $x \in G_1$, and y^{-1} denotes the inverse element of $y \in G_2$, then $\varphi(x^{-1}) = \varphi(x)^{-1}$: $e_2 = \varphi(e_1) = \varphi(x *_1 x^{-1}) = \varphi(x) *_2 \varphi(x^{-1})$, and similarly, $e_2 = \varphi(x^{-1}) *_2 \varphi(x)$. Thus $\varphi(x^{-1}) = \varphi(x)^{-1}$.

Example A.16 Let G_1 and G_2 be groups. Then the map $\varphi\colon G_1 \to G_2$ given by

$$\varphi(x) = e_2,$$

where e_2 is the identity element in G_2, is a homomorphism: $\varphi(x) *_2 \varphi(y) = e_2 *_2 e_2 = e_2 = \varphi(x *_1 y)$ for all $x, y \in G_1$. The map φ is often referred to as the *trivial* homomorphism.

The following theorem says that the composition of two homomorphisms is also a homomorphism.

Theorem A.17 *Let G_1, G_2 and G_3 be groups. If $\varphi_1\colon G_1 \to G_2$ and $\varphi_2\colon G_2 \to G_3$ are homomorphisms, then the composition $\varphi_2 \circ \varphi_1\colon G_1 \to G_3$ is also a homomorphism.*

Proof. Assume that all three groups are written multiplicatively. Then

$$\varphi_2(\varphi_1(xy)) = \varphi_2(\varphi_1(x)\varphi_1(y)) = \varphi_2(\varphi_1(x))\,\varphi_2(\varphi_1(y))$$

for all $x, y \in G_1$. Hence, $\varphi_2 \circ \varphi_1$ is a homomorphism. □

The following theorem describes how homomorphisms provides two subgroups.

Theorem A.18 *Let G_1 and G_2 be groups, and let $\varphi\colon G_1 \to G_2$ be a homomorphism. Then*

(1) *the* kernel *of φ, $\ker\varphi = \{x \in G_1 \mid \varphi(x) = e_2\}$, where e_2 is the identity element in G_2, is a normal subgroup of G_1;*

(2) *the* image *of φ, $\operatorname{im}\varphi = \{y \in G_2 \mid \text{there is an } x \in G_1 \text{ such that } \varphi(x) = y\}$ is a subgroup of G_2.*

Proof. Assume that G_1 and G_2 are both written multiplicatively. We prove part (1). If x and y are elements in $\ker\varphi$, then, since φ is a homomorphism, we have

$$\varphi(xy) = \varphi(x)\varphi(y) = e_2$$

where we have used the fact that $e_2 e_2 = e_2$. Since φ is assumed to be a homomorphism, we have $\varphi(e_1) = e_2$ where e_1 is the identity element in G_1. Hence, $e_1 \in \ker\varphi$. Finally, since $\varphi(x^{-1}) = \varphi(x)^{-1}$ for each $x \in G_1$, we have for each $x \in \ker\varphi$ that $x^{-1} \in \ker\varphi$. Thus $\ker\varphi$ is a group, and moreover, a subgroup of G_1. We now prove that $\ker\varphi$ is a normal subgroup of G_1. Let $x \in \ker\varphi$. Then for each $g \in G_1$, we have

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g^{-1}) = \varphi(g)e_2\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = e_2.$$

Hence, $\ker\varphi$ is a normal subgroup of G_1. This proves part (1).

We now prove part (2). Let x_1 and x_2 be two elements in G_1. Then, since φ is a homomorphism, we have

$$\varphi(x_1)\varphi(x_2) = \varphi(x_1 x_2) \in \operatorname{im}\varphi.$$

Since φ is a homomorphism, we have $e_2 \in \operatorname{im}\varphi$. To complete the proof of the fact that $\operatorname{im}\varphi$ is a subgroup of G_2, we use the fact that $\varphi(x^{-1}) = \varphi(x)^{-1}$ for each $x \in G_1$. Thus $\varphi(x)^{-1} \in \operatorname{im}\varphi$ for each $x \in G_1$. Hence, for each $y \in \operatorname{im}\varphi$, we have $y^{-1} \in \operatorname{im}\varphi$. This proves that $\operatorname{im}\varphi$ is a group, and moreover, that it is a subgroup of G_2. Hence, we have proved part (2). □

An injective homomorphism is said to be a *monomorphism* while a surjective homomorphism is said to be an *epimorphism*.

Definition A.19 **(Isomorphisms)** Let G_1 and G_2 be groups. We say that a bijective homomorphism $\varphi\colon G_1 \to G_2$ is an *isomorphism*.

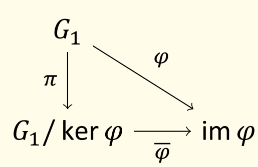
Thus a homomorphism which is both a monomorphism and an epimorphism is an isomorphism.

Example A.20 Let G be a group. The identity map $\mathrm{id}\colon G \to G$ is an isomorphism. In general, we refer to an isomorphism from a group G to itself as an *automorphism*.

Theorem A.21 **(Fundamental homomorphism theorem)** *Let G_1 and G_2 be groups, and let $\varphi\colon G_1 \to G_2$ be a homomorphism. Then the map $\overline{\varphi}\colon G_1/\ker\varphi \to \operatorname{im}\varphi$ given by*

$$\overline{\varphi}(x\ker\varphi) = \varphi(x)$$

is an isomorphism. Furthermore, if $\pi\colon G_1 \to G_1/\ker\varphi$ is the map given by $\pi(x) = x\ker\varphi$, then π is a homomorphism and $\varphi(x) = (\overline{\varphi}\circ\pi)(x)$ for each $x \in G_1$.



Proof. We first show that $\overline{\varphi}$ is well-defined. Let x_1 and x_2 be two elements in G. If $x_1\ker\varphi = x_2\ker\varphi$, we must show that $\overline{\varphi}(x_1\ker\varphi) = \overline{\varphi}(x_2\ker\varphi)$. In other words, we must show that $\varphi(x_1) = \varphi(x_2)$. Since $x_2 \in x_1\ker\varphi$, there is an element z in $\ker\varphi$ such that $x_2 = x_1 z$. Thus if e_2 is the identity element in G_2, we have

$$\varphi(x_2) = \varphi(x_1 z) = \varphi(x_1)\varphi(z) = \varphi(x_1)e_2 = \varphi(x_1)$$

where we have used the fact that φ is a homomorphism. Hence, $\overline{\varphi}$ is a well-defined map.

We now show that $\overline{\varphi}$ is an isomorphism, where we first show that it is a homomorphism, and then that it is bijective. By definition for $x_1\ker\varphi, x_2\ker\varphi \in G/\ker\varphi$, we have

$$\begin{aligned}\overline{\varphi}((x_1\ker\varphi)(x_2\ker\varphi)) &= \overline{\varphi}((x_1x_2)\ker\varphi)\\ &= \varphi(x_1x_2)\\ &= \varphi(x_1)\varphi(x_2)\\ &= \overline{\varphi}(x_1\ker\varphi)\overline{\varphi}(x_2\ker\varphi).\end{aligned}$$

Thus $\overline{\varphi}$ is a homomorphism.

Now assume that $\overline{\varphi}(x_1\ker\varphi) = \overline{\varphi}(x_2\ker\varphi)$. We must show that this implies that $x_1\ker\varphi = x_2\ker\varphi$. If $\overline{\varphi}(x_1\ker\varphi) = \overline{\varphi}(x_2\ker\varphi)$, then by definition, we have $\varphi(x_1) = \varphi(x_2)$. This implies that

$\varphi(x_2)^{-1}\varphi(x_1) = e_2$. Since φ is a homomorphism, we have $\varphi(x_2^{-1}x_1) = e_2$. Thus $x_2^{-1}x_1 \in \ker\varphi$, and so, $x_2^{-1}x_1 \ker\varphi = \ker\varphi$. Hence, $x_1 \ker\varphi = x_2 \ker\varphi$. Thus $\overline{\varphi}$ is injective. For any $\varphi(x) \in \operatorname{im}\varphi$, we have, by definition, that $\overline{\varphi}(x\ker\varphi) = \varphi(x)$. Hence, $\overline{\varphi}$ is surjective. Thus $\overline{\varphi}$ is bijective, and hence, it is an isomorphism.

By definition, we have, for elements x_1 and x_2 in G_1, that

$$\pi(x_1x_2) = (x_1x_2)\ker\varphi = (x_1\ker\varphi)(x_2\ker\varphi).$$

Hence, π is a homomorphism. Furthermore,

$$(\overline{\varphi}\circ\pi)(x) = \overline{\varphi}(\pi(x)) = \overline{\varphi}(x\ker\varphi) = \varphi(x)$$

for each $x \in G_1$. This completes the proof. □

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