Trial exam

Spring 2020

1 Let $X = \{a, b, c, d\}$.

Which of the following collections of subsets of X is *not* a topology on X?

Select one alternative:

- ☐ $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, d\}, \{a, d\}, \{d\}, X\}$
- ☐ $\mathcal{T} = \{\emptyset, \{a\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, X\}$
- ☐ $\mathcal{T} = \{\emptyset, \{b, c, d\}, \{d\}, X\}$
- ☐ $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$

Solution: The collection

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, X\}$$

is *not* a topology on X as T3 is not satisfied: the intersection of the open sets $\{a, b, c\}$ and $\{b, c, d\}$ is not an open set as $\{a, b, c\} \cap \{b, c, d\} = \{b, c\} \notin \mathcal{T}$. (Similarly, $\{a, d\} \cap \{b, c, d\} \notin \mathcal{T}$.)

2 Let $X = \{a, b, c, d, e\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, \{a, b, c\}, \{b, c\}, X\}$ be a topology on X.

Which of the following statements is *false*?

Select one alternative:

- ☐ The interior of $\{d, e\}$, $\text{Int}(\{d, e\})$, is equal to the empty set $\emptyset$.
- ☐ The closure of $\{e\}$, $\overline{\{e\}}$, is equal to $\{d, e\}$.
- ☐ X is compact.
- ☐ X is Hausdorff.

Solution: Since the points b and c are not separated by disjoint open sets, X is *not* Hausdorff. (The same argument applies to a and d, a and e, b and d, b and e, c and d, c and e, and d and e.)

3 Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Furthermore, let A be a subset of X.

Show that $x \in \overline{A}$ if and only if $B \cap A \neq \emptyset$ for every basis element $B \in \mathcal{B}$ where $x \in B$.

(Here $\overline{A}$ denotes the closure of A.)

Solution: This is essentially the same problem as Exercise 3.6 in lecture notes. We first show that $x \notin \overline{A}$ if and only if there is no neighborhood U of x such that $U \cap A = \emptyset$.

Let $x \notin \overline{A}$. Then $x \in \overline{A}^c = X \setminus \overline{A}$. Since $\overline{A}^c$ is open in X, $\overline{A}^c$ is a neighborhood of x such that $\overline{A}^c \cap A = \emptyset$.

For the other implication assume that there is a neighborhood U of x such that $U \cap A = \emptyset$. Since U is open in X, U^c is closed in X, and $A \subseteq U^c$. Thus as the closure of A is the smallest closed subset of X that contains A, it follows that $\overline{A} \subset U^c$. Hence, $x \notin \overline{A}$.

From what we have just proved it follows that $x \in \overline{A}$ if and only if there is a neighborhood U of x such that $U \cap A \neq \emptyset$. Since each basis element $B \in \mathcal{B}$ is open in X, and for each neighborhood U of x there is a basis element $B \in \mathcal{B}$ such that $x \in \mathcal{B} \subseteq U$, we have $x \in \overline{A}$ if and only if there is a basis element $B \in \mathcal{B}$ such that $B \cap A \neq 0$ and $x \in B$.

4 Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology, and consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$.

Show that the subset $A = \{x \in \mathbb{Q} \mid -\sqrt{5} < x < \sqrt{5}\}$ of $\mathbb{Q}$ is both open and closed in $\mathbb{Q}$.

Solution: Since the open interval $(-\sqrt{5}, \sqrt{5})$ is open in $\mathbb{R}$ and $A = (-\sqrt{5}, \sqrt{5}) \cap \mathbb{Q}$, it follows that A is open in the subspace topology on $\mathbb{Q}$. Similarly, since the closed interval $[-\sqrt{5}, \sqrt{5}]$ is closed in $\mathbb{R}$ and $A = [-\sqrt{5}, \sqrt{5}] \cap \mathbb{Q}$, it follows that A is closed in the subspace topology on $\mathbb{Q}$.

5 Let $X = \{a, b, c, d\}$ be given the topology $\mathcal{T}_X = \{\emptyset, \{a\}, \{a, c, d\}, \{c, d\}, X\}$, and let $Y = \{1, 2, 3\}$ be given the topology $\mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 3\}, Y\}$.

Find a basis for the product topology on $X \times Y$ expressed using bases for the topologies on X and Y, respectively.

Solution: Note that $\mathcal{B}_X = \{\{a\}, \{c, d\}, X\}$ is a basis for $\mathcal{T}_X$, and that $\mathcal{B}_Y = \{\{1\}, \{1, 3\}, Y\}$ is a basis for $\mathcal{T}_Y$. Then by Theorem 5.8 in the lecture notes, it follows that

$$\mathcal{B}_{X \times Y} = \{B_X \times B_Y \mid B_X \in \mathcal{B}_X \text{ and } B_Y \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$.

6 Let X be a topological space, and consider $I = [0, 1]$ as a subspace of $\mathbb{R}$ where $\mathbb{R}$ is given the standard topology. Furthermore, let the *cone on* X be the quotient space $CX = X \times I/\sim$, where $\sim$ is the equivalence relation on the product space $X \times I$ given by $(x, 0) \sim (x', 0)$ for all $x, x' \in X$.

Show that CX is path connected.

Solution: Denote by $[x, t]$ the equivalence class of (x, t), and let v denote the vertex of CX, i.e., v is the point corresponding to $[X, 0]$. Then for any $x \in X$ and any $t > 0$, there is a path $f : I \to X \times I$ given by

$$f(s) = (x, (1 - s)t).$$

In other words, f is a path in $X \times I$ from (x, t) to $(x, 0)$. Thus $\pi \circ f : I \to CX$ is a path in CX from $[x, t]$ to v. Hence, CX is path connected.

7 Let X be a topological space, and let $A_1, A_2, \dots, A_n$ be subspaces of X each of which is compact in X.

Show that $\displaystyle\bigcup_{i=1}^{n} A_i$ is compact in X.

Solution: Let $\mathcal{A}$ be an open cover of $\bigcup_{i=1}^{n} A_i$, i.e.,

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{U \in \mathcal{A}} U.$$

We must show that $\mathcal{A}$ has a finite subcover.

Note that $A_i \subseteq \bigcup_{U \in \mathcal{A}} U$ for all $i \in \{1, 2, \dots, n\}$. Hence, $\mathcal{A}$ is an open cover of each A_i. Since each A_i is compact in X, there is a finite subcover $\mathcal{A}_i$ of $\mathcal{A}$ that covers A_i. Thus

$$\bigcup_{i=1}^{n} \mathcal{A}_i$$

is a finite subcover of the open cover $\mathcal{A}$. Hence, $\bigcup_{i=1}^{n} A_i$ is compact in X.

8 Let X be a contractible space.

Show that X is path connected.

Solution: Let $a, b \in X$. Since X is contractible there is a homotopy $H\colon X \times I \to X$ such that

$$H(x,0) = x$$

and

$$H(x,1) = c$$

for all $x \in X$ for some fixed point $c \in X$. Then the map $\alpha\colon I \to X$ given by $\alpha(t) = H(a,t)$ is a path in X from $\alpha(0) = a$ to $\alpha(1) = c$. Similarly, the map $\beta\colon I \to X$ given $\beta(t) = H(b,t)$ is a path in X from $\beta(0) = b$ to $\beta(1) = c$. Thus a, b and c are all joined by paths. Hence, X is path connected.

9 Let B be a Hausdorff space, and let E be a topological space.

Show that if $p\colon E \to B$ is a covering map, then E must be Hausdorff.

Solution: Let e and e' be two distinct points in E, i.e., $e \neq e'$. If $p(e) \neq p(e')$, there are, by assumption that B is Hausdorff, disjoint open subsets U and U' in X such that $p(e) \in U$ and $p(e') \in U'$. Then $p^{-1}(U)$ and $p^{-1}(U')$ are disjoint neighborhoods of e and e', respectively.

Let U be a neighborhood of $p(e)$ that is evenly covered by p, i.e.,

$$p^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} V_\lambda$$

where V_λ is open in E, and $p|_{V_\lambda}\colon V_\lambda \to U$ is a homeomorphism for each $\lambda \in \Lambda$. If $p(e) = p(e')$, then there are distinct indices λ_1 and λ_2 in Λ such that $e \in V_{\lambda_1}$ and $e' \in V_{\lambda_2}$.

Hence, E is Hausdorff.

10 Let n be an integer that is greater than or equal to 3, and let $D^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq 1\}$ be considered as a subspace of $\mathbb{R}^n$ where $\mathbb{R}^n$ is given the standard topology.

Show that the inclusion map $i\colon D^n \setminus \{0\} \to D^n$ induces an isomorphism of fundamental groups. (Here 0 denotes the origin in $\mathbb{R}^n$.)

You may assume as a known fact that the m-sphere S^m is simply connected where S^m is considered as a subspace of $\mathbb{R}^{m+1}$ and m is an integer that is greater than or equal to 2.

Solution: Since D^n is contractible, and $D^n \setminus \{0\}$ is a deformation retract of S^{n-1} (the homotopy $H\colon D^n \setminus \{0\} \times I \to D^n \setminus \{0\}$ given by

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}, \qquad x = (x_1, x_2, \dots, x_n), \quad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

is a deformation retraction of $D^n \setminus \{0\}$ onto S^{n-1}), it follows that for any $d_0 \in D^n \setminus \{0\}$, we have

$$\pi_1(D^n \setminus \{0\}, d_0) \cong 0$$

and

$$\pi_1(D^n, d_0) \cong 0$$

where 0 denotes the trivial group and where we have used the fact that S^{n-1} is simply connected. Hence, the induced homomorphism $i_*\colon \pi_1(D^n \setminus \{0\}, d_0) \to \pi_1(D^n, d_0)$ is an isomorphism for each $d_0 \in D^n \setminus \{0\}$.