Exercises

7.1 Let $H_1 : f \simeq f'$ and $H_2 : g \simeq g'$. Then $H_3 : X \times I \longrightarrow Z$ given by

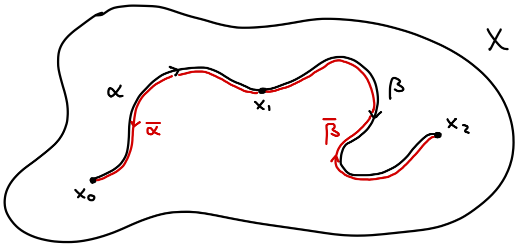
$$H_3(x,t) = H_2(H_1(x,t),t)$$

is a homotopy from $g \circ f$ to $g' \circ f'$: H_3 is clearly continuous, and

$$H_3(x,0) = H_2(H_1(x,0),0) = H_2(f(x),0) = g(f(x)) = (g \circ f)(x)$$
$$H_3(x,1) = H_2(H_1(x,1),1) = H_2(f'(x),1) = g'(f'(x)) = (g' \circ f')(x)$$

for all $x \in X$.

7.2 For $[f] \in \pi_1(X, x_0)$, we have



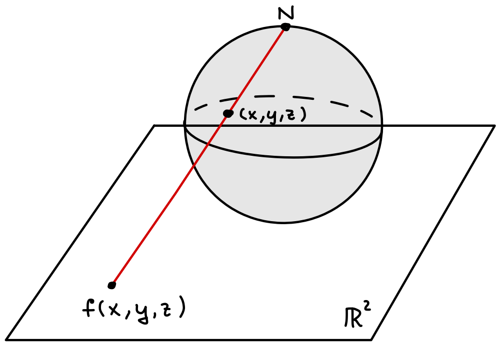
$$\hat{\alpha}([f]) = [\bar{\alpha} * f \, \alpha] = [\alpha]^{-1} * [f] * [\alpha]$$

such that if $\gamma = \alpha * \beta$, then

$$\hat{\gamma}([f]) = [\bar{\gamma} * f * \gamma] = [\overline{\alpha * \beta} * f * \alpha * \beta] = [\bar{\beta} * \bar{\alpha} * f * \alpha * \beta] = \hat{\beta}([\bar{\alpha} * f * \alpha]) = \hat{\beta}(\hat{\alpha}([f])).$$

Hence, $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

7.3 Let $N = (1,0,0) \in \mathbb{R}^3$. The map $f : S^2 \setminus N \longrightarrow \mathbb{R}^2$ given by



$$f(x,y,z) = \frac{(x,y)}{1-z}$$

is a continuous bijection with $f^{-1} : \mathbb{R}^2 \longrightarrow S^2 \setminus N$ given by

$$f^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2} \right).$$

As f^{-1} is also continuous, it follows that f is a homeomorphism (called stereographic projection). Since $\mathbb{R}^2$ is simply connected, it follows that $S^2 \setminus N$ is also simply connected. Hence,

$$\pi_1(S^2 \setminus N, s) \cong 0$$

where 0 denotes the trivial group.

7.4 Let $r : X \longrightarrow A$ be a retraction of X onto A. We want to show that A is closed.

Note that $A = \{x \in X \mid r(x) = x\}$. Let $\Delta = \{(x,x) \mid x \in X\}$ be the diagonal in $X \times X$, and let $f : X \longrightarrow X \times X$ be the map given by $f(x) = (x, r(x))$. Then f is clearly continuous. Since Δ is closed in $X \times X$ and $f^{-1}(\Delta) = \{x \in X \mid f(x) = (x, r(x)) \in \Delta\} = \{x \in X \mid x = r(x)\} = A$, it follows by continuity of f that A is closed in X.

7.5 Reflexivity and symmetry of homotopy equivalences is obvious.

If $X \simeq Y$ and $Y \simeq Z$, we must show that $X \simeq Z$. Assume that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homotopy equivalences with homotopy inverses $f' : Y \longrightarrow X$ and $g' : Z \longrightarrow Y$, respectively.

We will show that $g \circ f : X \to Z$ is a homotopy equivalence with homotopy inverse $f' \circ g' : Z \to X$. In other words, we show that

$$(g \circ f) \circ (f' \circ g') \simeq \mathrm{id}_Z \quad \text{and} \quad (f' \circ g') \circ (g \circ f) \simeq \mathrm{id}_X$$

where id is the identity map.

Since composition is associative, we have $(g \circ f) \circ (f' \circ g') = g \circ (f \circ f') \circ g'$. By assumption, $f \circ f' \simeq \mathrm{id}_Y$, and so, by Exercise 7.1, we have

$$g \circ (f \circ f') \simeq g \circ \mathrm{id}_Y .$$

As $g \circ \mathrm{id}_Y \simeq g$, it follows that

$$g \circ (f \circ f') \circ g' \simeq g \circ \mathrm{id}_Y \circ g' \simeq g \circ g' \simeq \mathrm{id}_Z .$$

A completely analogous argument shows that $(f' \circ g') \circ (g \circ f) \simeq \mathrm{id}_X$.

Hence, the relation of homotopy equivalence is an equivalence relation on any set of topological spaces.

$\boxed{7.6}$ Assume that X is contractible, i.e., $\mathrm{id}_X \simeq c_{x_0}$ where $c_{x_0} : X \to X$ is the constant map at $x_0 \in X$:

$$c_{x_0}(x) = x_0$$

for all $x \in X$. Let $Y = \{x_0\}$, and let $i : Y \to X$ be the inclusion map. Then $c_{x_0} \circ i = \mathrm{id}_Y$, and so, $c_{x_0} \circ i \simeq \mathrm{id}_Y$. Since $i \circ c_{x_0} = c_{x_0}$ and $c_{x_0} \simeq \mathrm{id}_X$, it follows that $i \circ c_{x_0} \simeq \mathrm{id}_X$. Hence, $X \simeq Y$. (Here we have thought of c_{x_0} as a map from X to Y.)

Now assume that $X \simeq Y$ where $Y = \{c\}$. Then there is a homotopy equivalence $f : X \to Y$ with homotopy inverse $g : Y \to X$. Hence, $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Since $Y = \{c\}$, we have

$$g(c) = x_*$$

for some $x_* \in X$. Then for all $x \in X$, we have

$$(g \circ f)(x) = g(f(x)) = g(c) = x_* .$$

This means that $g \circ f = c_{x_*}$, i.e., $g \circ f$ is a constant map. Hence, $\mathrm{id}_X \simeq g \circ f \simeq c_{x_*}$. Thus X is contractible.

$\boxed{8.1}$ Since the map $p : \mathbb{R} \to S^1$ given by

$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

is a covering map, and the identity map $\mathrm{id}_{S^1} : S^1 \to S^1$ is a covering map, it follows by the fact that the product of two covering maps is a covering map that

$$p \times \mathrm{id}_{S^1} : \mathbb{R} \times S^1 \longrightarrow S^1 \times S^1 = T^2$$

is a covering map. Hence, $\mathbb{R} \times S^1$ is a covering space of the torus $T^2 = S^1 \times S^1$.

$\boxed{8.2}$ Since E is path connected, the lifting correspondence

$$\Phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is surjective where $p(e_0) = b_0$. Furthermore, since B is simply connected, $p^{-1}(b_0)$ consists of only one point, e_0.

We must show that p is a bijection. In particular, we will show that $p^{-1}(b)$ consists of only one point for every $b \in B$.

Let $C = \{b \in B \mid |p^{-1}(b)| = 1\}$ and $D = \{b \in B \mid |p^{-1}(b)| \neq 1\}$. Note that $C \cap D = \emptyset$ and $C \cup D = B$. We will show that C and D are both open in B. If $b \in C$ then there is a neighborhood $\mathcal{U}$ of b that is evenly covered by p, i.e., $p^{-1}(\mathcal{U}) = V$ where $p|_V : V \to \mathcal{U}$ is a homeomorphism. Thus for $b' \in \mathcal{U}$, $|p^{-1}(b')| = 1$. Hence, $b \in \mathcal{U} \subseteq C$, and so, C is open in B. Similarly, if $b \in D$ there is a neighborhood $\mathcal{U}$ of b that is evenly covered by p. Then $p^{-1}(\mathcal{U}) = \coprod_i V_i$ where $|V_i| = |p^{-1}(b)|$ and $p|_{V_i} : V_i \to \mathcal{U}$ is a homeomorphism for all i. Thus for $b' \in \mathcal{U}$, $|p^{-1}(b')| = |p^{-1}(b)|$. Hence, $b \in \mathcal{U} \subseteq D$, and so, D is open D in B.

Since B is simply connected, it is also connected. As $b_0 \in C$ and both C and D are open in B, we have by connectivity of B that $B = C$ and $D = \emptyset$. Hence, p is a bijection.

By the fact that every covering map is also an open map, and that a bijective open map is a homeomorphism, we conclude that $p : E \to B$ is a homeomorphism.

(To see that every covering map is an open map, let $p : E \to B$ be a covering map and let V be an open subset of E. If $x \in p(V)$, then we can choose a neighborhood $\mathcal{U}$ of x that is evenly covered by p. Assume that $p^{-1}(\mathcal{U}) = \coprod_\lambda V_\lambda$. Then there is a point $y \in V$ such that $p(y) = x$ and $y \in V_{\lambda'}$ for $\lambda' \in \Lambda$. Then $V \cap V_{\lambda'}$ is open in $V_{\lambda'}$. Since $p|_{V_{\lambda'}} : V_{\lambda'} \to \mathcal{U}$ is a homeomorphism, it follows that $p(V \cap V_{\lambda'})$ is open in $\mathcal{U}$, and hence, it is open in B. Thus $p(V \cap V_{\lambda'})$ is a neighborhood of x where $p(V \cap V_{\lambda'}) \subseteq p(V)$. Hence, $p(V)$ is open, and thus p is an open map.)

$\boxed{8.3}$ Let $p : E \to B$ be a covering map where $p(e_0) = b_0$ for $e_0 \in E$.

Assume that $\tilde{f} : I \to E$ is a loop in E based at e_0, and that

$$p_*([\tilde{f}]) = [p \circ \tilde{f}]$$

is the identity element of $\pi_1(B, b_0)$. Then there is a path homotopy $H : p \circ \tilde{f} \simeq_p c_{b_0}$ where $c_{b_0} : I \longrightarrow B$ is the constant path in B at b_0. If $\tilde{H}$ is the lifting of H to E such that $\tilde{H}(0,0) = e_0$, then $\tilde{H} : \tilde{f} \simeq_p c_{e_0}$ where $c_{e_0} : I \to E$ is the constant path in E at e_0. Hence, $[\tilde{f}]$ is the identity element of $\pi_1(E, e_0)$, and so, $p_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$ is a monomorphism.

$\boxed{8.4}$ Recall that $\mathbb{RP}^n = S^n/\sim$ where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x = \pm y$, i.e., $[x] = \{-x, x\}$. Let $q : S^n \longrightarrow \mathbb{RP}^n$ be the quotient map given by $q(x) = [x]$.

We will show that q is a covering map. Let $a : S^n \longrightarrow S^n$ be the map given by $a(x) = -x$ (we call it the antipodal map). Clearly, a is a homeomorphism and $a^2 = a \circ a = id_{S^n}$. Note that for $[x] \in \mathbb{RP}^n$, we have $[x] = \{x, a(x)\}$. We first show that q is an open map. If $\mathcal{U}$ is an open subset of S^n then

$$q^{-1}(q(\mathcal{U})) = \mathcal{U} \cup a(\mathcal{U}).$$

Since a is a homeomorphism, we must have that $a(\mathcal{U})$ is open in S^n. Hence, $\mathcal{U} \cup a(\mathcal{U})$ is open in S^n. Thus $q(\mathcal{U})$ is open in $\mathbb{RP}^n$ by definition of the quotient topology. Hence, q is an open map.

We now show that q is a covering map. Choose an $x \in S^n$. Since $a(x) \neq x$ and S^n is Hausdorff, there are neighborhoods $\mathcal{U}_1$ of x and $\mathcal{U}_2$ of $a(x)$ such that $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$. Let $\mathcal{U} = \mathcal{U}_1 \cap a(\mathcal{U}_2)$. Then $\mathcal{U}$ is a neighborhood of x as $x \in \mathcal{U}$ and $\mathcal{U}$ is open in S^n. Furthermore, we have $\mathcal{U} \cap a(\mathcal{U}) = \emptyset$. Since q is an open map, $q(\mathcal{U})$ is a neighborhood of $[x] \in \mathbb{RP}^n$ where $q^{-1}(q(\mathcal{U})) = \mathcal{U} \cup a(\mathcal{U})$. Thus $q|_{\mathcal{U}} : \mathcal{U} \to q(\mathcal{U})$ and $q|_{a(\mathcal{U})} : a(\mathcal{U}) \to q(a(\mathcal{U})) = q(\mathcal{U})$ are continuous bijective maps. Since q is also open, it follows that these maps are in fact homeomorphisms. Hence, $q(\mathcal{U})$ is a neighborhood of $[x]$ that is evenly covered by q, and so, q is a covering map. (It is a 2-fold covering.)

Finally, since S^n for $n \geq 2$ is simply connected, we have by the lifting correspondence a bijection

$$\Phi : \pi_1(\mathbb{RP}^n, x_0) \longrightarrow q^{-1}(x_0)$$

for any $x_0 \in \mathbb{RP}^n$. As $|q^{-1}(x_0)| = 2$ for all $x_0 \in \mathbb{RP}^n$, it follows that

$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}/2.$$