General Topology MA3002 Spring 2016

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Contents

Introduction

Let us start with an arguable opinion.

> *General topology is not so much a theory of topological spaces as it is a theory of continuous maps.*

For example, we will mostly study topological spaces by studying continuous maps into or out of them. For this reason, we can and will in Chapter I dwell on continuous maps between metric spaces for a while before actually defining topological spaces.

Even more drastically, some topological spaces that we will meet shall be characterized by the continuous maps into or out of them. This point of view will be the main theme in Chapter II.

In Chapter III we will see that two of the most prominent properties of topological spaces, the Hausdorff property and compactness, are actually better understood when they are thought of as specializations of properties of continuous maps.

Relatedly, but on the other hand, continuous maps can be investigated through their fibers, which are topological spaces. Later in Chapter V we will also meet mapping spaces. These topological spaces are designed to study continuous maps as well.

And what has Chapter IV to do with this? This question is actually somewhat fun to answer, and we will think about it when the time has come.

What is missing from these notes?

Later in some lecture we worked out a natural bijection between the set of open subsets of a topological space and the set of continuous maps into the two-point space with three open subsets.

Examples of compact Hausdorff spaces that were done in class but are not spelled out in this text: Cantor space as an infinite product and the p-adic numbers as a subspace of an infinite product.

Chapter I

Continuous maps

1 Continuous maps between metric spaces

In this section we will see various reformulations of the notion of a continuous map between metric spaces.

1.1 Metric spaces

Definition 1.1. A *metric space* is a set M together with a function

$$d\colon M \times M \longrightarrow \mathbb{R}$$

that satisfies the following properties.

We have $d(p,q) \geqslant 0$ for all p,q, and $d(p,q) = 0$ if and only if $p = q$.
We have $d(q,p) = d(p,q)$ for all p,q.
The triangle inequality

$$d(p,r) \leqslant d(p,q) + d(q,r)$$

is satisfied for all p,q,r.

Example 1.2. Take $\mathbb{R}$ with the metric $d(p,q) = |p-q|$. Or $\mathbb{R}^2$ with the euclidean metric. If we use the set $\mathbb{C}$ of complex numbers to model the plane $\mathbb{R}^2$, then the metric is given my the same formula.

Example 1.3. If d is a metric on M, and $S \subseteq M$ is any subset, then d is also a metric on S.

Example 1.4. Take M to be any set and $d(p,q) = 1$ if $p \neq q$. This is the *discrete metric* on M.

Example 1.5. Take $M = \mathbb{Z}$ and set $d(a,b) = (1/2)^n$ if $a \neq b$ and 2^n is the highest power of 2 that divides $a-b$. This is the 2-*adic metric* on $\mathbb{Z}$. The number 2 can be replaced by any prime number.

1.2 Continuous maps between them

Definition 1.6. A map $f\colon M \to N$ between metric spaces is *continuous* if for all q in N and all $\varepsilon > 0$ and all p in M such that $f(p) = q$ there is a $\delta > 0$ such that $d_M(x,p) < \delta$ implies $d_N(fx,q) < \varepsilon$.

Example 1.7. Identity maps $M \to M, p \mapsto p$ are continuous ($\delta = \varepsilon$).

Example 1.8. Constant maps are continuous (all $\delta > 0$ will do).

Example 1.9. If the metric on M is discrete, then any function $f\colon M \to N$ is continuous.

Example 1.10. The function $f\colon \mathbb{R} \to \mathbb{R}$ that sends $x \neq 0$ to $1/x$ and 0 to itself is not continuous.

Exercise 1. Let M be a metric space, and a be a point of M. Then the function

$$M \longrightarrow \mathbb{R}, b \mapsto d(a,b)$$

is continuous. Hint: First check that this follows from the inequality

$$|d(x,y) - d(x,z)| \leqslant d(y,z),$$

and then prove that inequality.

1.3 Balls

In topology, inequalities between numbers are replaced by inclusions between subsets.

Definition 1.11. A *ball* in a metric space M with metric d is a subset of the form

$$\mathrm{B}(p,\delta) = \{x \in M \,|\, d(x,p) < \delta\}.$$

Example 1.12. A ball does not determine a point p and a radius δ: In a discrete metric space M, we have $M = \mathrm{B}(p,\delta)$ in dependent of p and δ as long as $\delta > 1$.

Lemma 1.13. *If a point lies in a ball, then this ball contains a ball centered around this point.*

Proof. If q is in $\mathrm{B}(p,\delta)$, then $\varepsilon = \delta - d(p,q) > 0$, and the ball $\mathrm{B}(q,\varepsilon)$ centered at q is contained in the ball $\mathrm{B}(p,\delta)$. □

Proposition 1.14. *A map $f\colon M \to N$ between metric spaces is continuous if for all balls B in N and all p in M that map into B there is a ball A around p that is mapped into B.*

Proof. This is really just a reformulation, and uses the Lemma before. □

1.4 Pre-images and fibers

Definition 1.15. Let $f\colon A \to B$ be a map of sets. For a subset $S \subseteq B$, the subset

$$f^{-1}S = \{\, a \in A \mid f(a) \in S \,\}$$

is the *pre-image* of S under f. If $S = \{b\}$ for a single element b, then

$$f^{-1}\{b\} = \{\, a \in A \mid f(a) = b \,\}$$

is the *fiber* of f over b.

Note that f^{-1} here does not refer to the inverse function $f^{-1}\colon B \to A$, because most of the time, such an inverse will not exits. When it exits, there is no ambiguity.

Example 1.16. Given a surface in $\mathbb{R}^3$, for example a torus, we can study the fibers of the height function $(x,y,z) \mapsto z$.

If $\mathcal{P}_A$ denotes the set of all subsets of A, then we have a map

$$\begin{aligned} \mathcal{P}_f\colon \mathcal{P}_B &\longrightarrow \mathcal{P}_A \\ S &\longmapsto \mathcal{P}_f(S) = f^{-1}S. \end{aligned}$$

Exercise 2. Fun fact: The map f is injective if and only if the map $\mathcal{P}_f$ is surjective, and the map f is surjective if and only if the map $\mathcal{P}_f$ is injective. It is a good exercise in pre-images to show that.

Proposition 1.17. *A map between metric spaces is continuous if pre-images of balls are unions of balls.*

Proof. Again, this is really just a reformulation. □

1.5 Open subsets

Definition 1.18. A subset of a metric space M is *open in M* if it is a union of balls.

Examples 1.19. Every ball is a union of balls, so balls are open in M. The converse is not true: The empty set Ø is a union of (no) balls, so that it is open in M, but it is not a ball itself, because balls are never empty.

Example 1.20. The metric space M is open in M, because

$$M = \bigcup_{p \in M} \bigcup_{\delta > 0} \mathrm{B}(p, \delta)$$

shows that it is a union of balls. On the other hand, the metric space M is a ball if and only if it is bounded.

Example 1.21. For every x in $\mathbb{R}$, the subset $\mathbb{R} \setminus x$ is open in $\mathbb{R}$: If y is not x, then the ball $\mathrm{B}(y, d(x,y))$ contains y but not x.

Example 1.22. For every x in $\mathbb{R}$, the subset $\{x\}$ is not open in $\mathbb{R}$: There is no ball that is contained in $\{x\}$. However, the subset $\{x\}$ is open in $\{x\}$ by the example above.

Proposition 1.23. *A map between metric spaces is continuous if pre-images of open subsets are open.*

Proof. One direction is clear, because balls are open. For the other one: If an open subset V is the union of balls B, then the pre-image $f^{-1}V$ is the union of the pre-images $f^{-1}B$. Therefore, if f is continuous, the pre-image $f^{-1}V$ will be a union of a union of balls. This is again a union of balls, and therefore open. □

2 Continuous maps between topological spaces

In this section we introduce the structure that is made to support continuous maps: topological spaces.

2.1 Topological spaces

Definition 2.1. A *topological space* is a set X together with a set $\mathcal{T}$ of subsets of X that are called *open in* X, such that the following properties are satisfied.

The subsets $\emptyset$ and X are open.
All unions of open subsets are open.
All finite intersections of open subsets are open.

We say (confusingly) that $\mathcal{T}$ is a *topology* on X.

Proposition 2.2. *If d is a metric on M, then the set of open subsets (as defined before) is a topology $\mathcal{T}(d)$ on M.*

Proof. The first two properties are easy to see. Given two open subsets U and V, we have to show that their intersection is open. If p is in the intersection, we need to find a ball that contains p and that lies in the intersection. There are balls that contain p and lie in U and V, respectively. By making them smaller, if necessary, we can assume that they are centered at p. Then their intersection in the one with the smaller radius, and that lies in both U and V. □

Example 2.3. Take the set $\mathbb{R}$ of real numbers with the usual metric/topology. Then the subsets $U_n = \mathrm{B}(0, 1/n)$ are open in $\mathbb{R}$, but their intersection $\{0\}$ is not. This explains the finiteness condition in the third property.

Example 2.4. The discrete metric d definies the *discrete topology*, where all subsets are open. This is so, because $\{p\} = \mathrm{B}(p, 1/2)$ is a ball, so that all subsets are unions of balls.

The discrete topology is the unique topology where the singletons are open.

The topologies on a given set X are partially ordered by inclusion. The discrete topology is obviously the largest one. There is also a smallest one.

Example 2.5. Given any set S, there is also the *indiscrete topology* $\{\emptyset, S\}$ on S. It is minimal. This does not come from a metric if S has at least two points, because in a metric space M, the subsets $M \setminus p$ are open. Thus, if M is also indiscrete, then $M = \{p\}$.

This examples indicates that the map $d \mapsto \mathcal{T}(d)$ from the set of metrics on X to the set of topologies on X does not have to be surjective. Neither does it have to be injective.

Example 2.6. If M has two elements, a metric on M is essentially a positive real number, the distance between the two points. But a finite set has only finitely many topologies on it.

More generally and specifically:

Example 2.7. If M is a finite metric space with metric d, then $\mathcal{T}(d)$ is always the discrete topology on X. Given any p in M, set

$$\delta = \min\{\, d(p,q) \,|\, q \neq p \,\}.$$

Then $\mathrm{B}(p, \delta) = \{p\}$ is open.

2.2 Continuous maps

We have seen that a map between metric spaces is continuous if and only if pre-images of open subsets are open. This motivates the following definition.

Definition 2.8. A map between topological spaces is *continuous* if and only if pre-mages of open subsets are open.

Examples 2.9. The definition has been made so that we already know lots of (and in a sense: all) continuous maps between metric spaces.

Examples 2.10. Constant maps are continuous, regardless of the topologies involved, because the pre-images are either empty or the entire space, and these are always open.

Proposition 2.11. *If $\mathcal{S}$ and $\mathcal{T}$ are two topologies on a set X, then the identity map $X \to X$, with $x \mapsto x$, is continuous as a map $(X, \mathcal{S}) \to (X, \mathcal{T})$ if and only if we have an inclusion $\mathcal{T} \subseteq \mathcal{S}$. In particular, the identity map is always continuous if the topology is the same on both sides.*

Proof. This follows from $\mathrm{id}^{-1}U = U$. $\square$

Proposition 2.12. *Compositions of continuous maps are continuous.*

Proof. This follows from $(gf)^{-1}W = f^{-1}g^{-1}W$. $\square$

Example 2.13. If X is discrete, then every map $X \to Y$ is continuous. In fact, this characterizes the discrete topology on X: If $\mathcal{T}$ is a topology on X such that all maps $f\colon X \to Y$ into all topological spaces Y are continuous, then $\mathcal{T}$ is the discrete topology. To see this, look at $Y = X$ with the discrete topology, and $f = \mathrm{id}$. This can only be continuous if $\mathcal{T}$ is discrete.

Example 2.14. If Y is indiscrete, then every map $X \to Y$ is continuous. In fact, this characterizes the indiscrete topology on Y.

2.3 Homeomorphisms

Definition 2.15. A continuous map $f\colon X \to Y$ is a *homeomorphism* if there is a continuous map $g\colon Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Example 2.16. Homeomorphism are continuous bijections, but the converse is not true. If X has at least two elements, then the identity map from X with the discrete topology to X with the indiscrete topology is a continuous bijection that is not a homeomorphism.

Example 2.17. It can happen that there are two different topologies on a set so that the resulting spaces are homeomorphic: Take $J = \{1,2\}$ with the two different topologies $\mathcal{T}_j = \{\emptyset, \{j\}, J\}$ for $j \in J$. Then the map that interchanges 1 and 2 is a homeomorphism $(J, \mathcal{T}_1) \cong (J, \mathcal{T}_2)$.

3 Closed subsets and related terminology

In this section we will introduce closed subsets and related terminology.

3.1 Closed subsets

Let X be a topological space.

Definition 3.1. A subset A of X is *closed in* X (with respect to the given topology) if and only if its complement $X \setminus A$ is open in X.

Clearly, from the definition, by passage to complements:

Proposition 3.2. *The subsets X and $\emptyset$ are closed. All intersections of closed subsets are closed. All finite unions of closed subsets are closed.*

Examples 3.3. If $X = \mathbb{R}$ with the usual topology, then: The subset $[\,0,1\,]$ is closed in $\mathbb{R}$, but not open in $\mathbb{R}$. The subset $]\,0,1\,[$ is open in $\mathbb{R}$, but not closed in $\mathbb{R}$. The subset $\mathbb{Q}$ is neither closed nor open in $\mathbb{R}$. The subsets $\emptyset$ and $\mathbb{R}$ are both closed and open in $\mathbb{R}$.

Examples 3.4. We will later see that there are the only two subsets of $\mathbb{R}$ that are both open and closed. In general, a topological space can have many subsets that are both open and closed. For example, in a discrete space, every subset is both open and closed.

Examples 3.5. We have seen that a topological space is discrete if and only if all singletons are open. However, in most topological spaces (for example, as we have seen, in all metric spaces) the singletons are closed, even if the space is far from being discrete.

Examples 3.6. If X is any set, finite or infinte. Say that a subset A is closed if A is finite or if $A = X$. Say that $U \subseteq X$ is open if $X \setminus U$ is closed. This defines a topology on X. The singletons are closed. The topology is discrete if and only if X is finite.

3.2 Closure

Let X be a topological space, and pick a subset S in X. Let $\mathcal{A}(S)$ denote the set of closed subsets in X that contain S. For example, we have $S \in \mathcal{A}(S)$ if and only if the subset S is closed in X.

Definition 3.7. The subset

$$\overline{S} = \bigcap_{A \in \mathcal{A}(S)} A$$

of X is called the *closure of S in X*.

Proposition 3.8. *The subset $\overline{S}$ is closed in X, and it contains S. If B is a closed set in X that contains S, then $\overline{S} \subseteq B$. In other words: The closure of S is the minimal closed subsets of X that contains S. In particular, we have $\overline{S} = S$ if and only if S is closed in X. And we have $\overline{\overline{S}} = \overline{S}$ for all S.*

Proof. The subset $\overline{S}$ is closed as the intersection of closed subsets. Since every of those subsets contains S, so does the intersection. If B is a closed set in X that contains S, then B is in $\mathcal{A}(S)$, so that it contains the intersection $\overline{S}$.

On the one hand, if $\overline{S} = S$, then S is closed in X, because $\overline{S}$ is.

On the other hand, if a subset S is closed in X, then it is the minimal closed subset of X that contains S, which implies $\overline{S} = S$.

The last sentence follows from this, because $\overline{S}$ is closed, so that $\overline{\overline{S}} = \overline{S}$. □

Exercise 3. A point p in X is contained in the closure of a subset S if and only if every open subset of X that contains p intersects S.

Examples 3.9. If $X = \mathbb{R}$ with the usual (metric) topology, then

$$\overline{]0,1[} = [0,1].$$

However, if $X = \mathbb{R}$ with the discrete (metric) topology, then

$$\overline{]0,1[} =]0,1[.$$

Examples 3.10. We have

$$\overline{\mathrm{B}(p,1)} \neq \{q \in M \,|\, d(p,q) \leqslant 1\}$$

for the discrete metric with at least two points. The ball on the left hand side is $\{p\}$, which is already closed, whereas the right hand side is M.

3.3 Dense subsets

Definition 3.11. The subset of a topological space X is called *dense in X* if its closure is X.

Example 3.12. For each topological space X, the subset X is dense in X. The empty subset $\emptyset$ is dense in X if and only if $X = \emptyset$ is empty. Every non-empty subset is dense in the indiscrete topology. This is actully another characterization of the indiscrete topology.

Exercise 4. Give a characterization of the discrete topology in terms of dense subsets? Hint: First find out which subsets are dense.

Example 3.13. The subset $\mathbb{Q}$ is dense in $\mathbb{R}$. This statement says that the only closed subset of the real numbers that contains all rational numbers is $\mathbb{R}$ itself. In other words, if we have an open subset that is contained in the set of irrational numbers, it is empty. Every non-empty open subset of $\mathbb{R}$ contains a rational number. Every ball in $\mathbb{R}$ contains a rational number.

3.4 More about continuous maps

Clearly, from the definition, by passage to complements: A map $f\colon X \to Y$ between topological spaces is continuous if and only if pre-images of closed subsets are closed.

Proposition 3.14. *A map $f\colon X \to Y$ between topological spaces is continuous if and only if*

$$f\left(\overline{S}\right) \subseteq \overline{f(S)}$$

for all subsets S in X.

Proof. Let us first assume that f is continuous. For every subset B of Y that is closed and contains $f(S)$, the pre-image $f^{-1}B$ is closed and contains S, so that also $\overline{S} \subseteq f^{-1}B$ holds. But then $f(\overline{S}) \subseteq B$. Taking the intersection of all B, we get the stated inclusion.

On the other hand, if there is a closed subset B in Y such that the pre-image $f^{-1}B$ is not closed in X, then $S = f^{-1}B$ is strictly contained in $\overline{S}$: Some point in $\overline{S}$ is not mapped into B. Since $f(S) \subseteq B$, and B is closed, we actually have $\overline{f(S)} \subseteq B$, and the same point in $\overline{S}$ is not mapped into $\overline{f(S)}$. $\square$

4 Generating topologies: bases and subbases

In this section we will contemplate a bit on the way we have obtained a topology from a metric. This involved an intermediate step, the set of balls.

4.1 Generating topologies

Recall that we have partially ordered the topologies on a given set X of points by inclusion. The minimal topology $\{\emptyset, X\}$ is the indiscrete one, and the maximal topology $\mathcal{P}_X$ is the discrete one.

Lemma 4.1. *Given a set of topologies on a set X, their intersection is also a topology on X.*

Example 4.2. It is not true that the union of two topologies has to be a topology again. Take $X = \{1,2,3\}$ with the topologies

$$\mathcal{T}_j = \{\emptyset, \{j\}, \{3\}, \{j,3\}, X\}$$

for $j = 1,2$. Then

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{P}_X \setminus \{1,2\}$$

is not a topology on X.

Given a set $\mathcal{G}$ of subsets of X, there is always a topology on X such that the elements of $\mathcal{G}$ are open with respect to that topology (take $\mathcal{P}_X$). The smallest one is this one:

Definition 4.3. The topology

$$\langle \mathcal{G} \rangle = \bigcap_{\substack{\mathcal{G} \subseteq \mathcal{T} \\ \mathcal{T} \text{ topology}}} \mathcal{T}$$

is the *topology generated by* $\mathcal{G}$.

Example 4.4. We have $\langle \emptyset \rangle = \{\emptyset, X\}$.

Example 4.5. If $\mathcal{G} = \{\{x\} \,|\, x \in X\}$, then $\langle \mathcal{G} \rangle = \mathcal{P}_X$.

Example 4.6. We have $\langle \mathcal{G} \rangle = \mathcal{G}$ if and only if $\mathcal{G}$ is a topology on X.

Exercise 5. Can you put a topology on the set of subsets of X such that $\langle \mathcal{G} \rangle$ is the closure of $\mathcal{G}$ with respect to this topology?

4.2 Bases and subbases

Definition 4.7. A set $\mathcal{S}$ of subsets of X is called a *subbasis* (for the topology $\langle\,\mathcal{S}\,\rangle$) if X is the union of $\mathcal{S}$.

$$X = \bigcup_{S \in \mathcal{S}} S$$

A subbasis $\mathcal{B}$ of subsets of X is called a *basis* (for the topology $\langle\,\mathcal{B}\,\rangle$) if, in addition, for all B_1 and B_2 in $\mathcal{B}$ and x in the intersection $B_1 \cap B_2$ there is B in $\mathcal{B}$ that contains x and is contained in $B_1 \cap B_2$.

Example 4.8. The set of balls is a basis for a topology on a metric space. We will see in a moment that this is the usual metric topology.

Example 4.9. The set of singletons is a basis for the discrete topology.

Example 4.10. The set $\{\,X\,\}$ is a basis for the indiscrete topology.

Proposition 4.11. *The topology generated by a basis $\mathcal{B}$ consists of the unions of subsets of $\mathcal{B}$.*

Proof. Let $\mathcal{U}$ be set of unions of subsets of $\mathcal{B}$. We want to show $\mathcal{U} = \langle\,\mathcal{B}\,\rangle$. The inclusion $\mathcal{U} \subseteq \langle\,\mathcal{B}\,\rangle$ is clear, because $\langle\,\mathcal{B}\,\rangle$ is a topology that contains $\mathcal{B}$, and then it also has to contain the unions of subsets of $\mathcal{B}$. The other inclusion $\langle\,\mathcal{B}\,\rangle \subseteq \mathcal{U}$ will follow once we know that $\mathcal{U}$ is a topology on X, because it contains $\mathcal{B}$, and $\langle\,\mathcal{B}\,\rangle$ is the minimal such. It therefore remains to be seen that $\mathcal{U}$ is a topology on X. This is done in the same way in which we showed that the unions of sets of balls give a topology on any metric space, and can be left as an exercise. □

Example 4.12. The set of balls is a basis for the usual metric topology on a metric space.

Proposition 4.13. *If $\mathcal{S}$ is a subbasis, the set of finite intersections of elements in $\mathcal{S}$ is a basis (for the same topology).*

Proof. If B_1 and B_2 are finite intersections of elements in $\mathcal{S}$, then so is $B = B_1 \cap B_2$, and this works for all x. □

Proposition 4.14. *Let X and Y be topological spaces, and let $\mathcal{S}$ be a (sub)basis for the topology on Y. Then a map $X \to Y$ is continuous if and only if pre-images of the elements in $\mathcal{S}$ are open in X.*

Proof. One direction is clear. The other one follows since taking pre-images commutes with unions and (finite) intersections. □

Chapter II

Universal properties

5 Subspaces

Topologies can be transported along maps. We will see how this works in this section and the next.

5.1 Induced topologies

Proposition 5.1. *Let Y be a topological space, and let $f\colon X \to Y$ be a map from a set X into Y. There is a unique topology on X such that a given map $t\colon T \to X$ is continuous if and only the composition $ft\colon T \to Y$ is continuous.*

Proof. First, we check that the set

$$\{\, f^{-1}V \mid V \text{ open in } Y \,\}$$

is a topology on X.

Secondly, we claim that this topology has the desired property: A given map $t\colon T \to X$ is continuous if and only the composition $ft\colon T \to Y$ is continuous. One direction is clear. For the other one, if U is open in X, we can write $U = f^{-1}V$ for some open V in Y. Then

$$t^{-1}U = t^{-1}f^{-1}V = (ft)^{-1}V,$$

so that, conversely, the continuity of ft implies that of t.

It remains to show uniqueness. For that, let us first assume that we have a topology $\mathcal{T}$ on our topological space X with the following property: A given map $t\colon T \to X$ from a given topological space T is continuous if and only the composition $ft\colon T \to Y$ is continuous. (For example, this topology $\mathcal{T}$ could be the one that we constructed above, but it does not have to be that one.) Then, taking t to be the identity map id_X of the topological space $(X, \mathcal{T})$, which is continuous, we see that f is continuous with respect to any such topology $\mathcal{T}$.

We can now complete the uniqueness argument. If $\mathcal{S}$ and $\mathcal{T}$ are two topologies on X with the stated property, then we can look at id_X as a map from $(X, \mathcal{T})$ to $(X, \mathcal{S})$. By the properties of the topologies, we see that this map id_X is continuous, so that $\mathcal{S} \subseteq \mathcal{T}$. By symmetry, we get $\mathcal{S} = \mathcal{T}$. □

Definition 5.2. This topology is called the topology *induced by f on X*.

5.2 Subspaces

Definition 5.3. Let X be a subset of the topological space Y. The topology on X induced by the inclusion $i\colon X \to Y, x \mapsto x$ is called the *subspace topology* on X.

Since $i^{-1}V = X \cap V$, the subspace topology is given as

$$\{X \cap V \mid V \text{ open in } Y\}.$$

Note that X is always open in X, whereas it does not have to be open in Y.

Exercise 6. Let $Y = \mathbb{R}$ with respect to the standard topology, and let $X = \mathbb{Z}$. Then the induced topology is discrete. If $X = \mathbb{Q}$, then the induced topology is not discrete (and not indiscrete).

Exercise 7. Let $Y = \mathbb{R}$ with respect to the standard topology, and let $X = [0,1[$ have the subspace topology, then $[0,1[$ is both open and closed in X, but clearly not in Y.

Exercise 8. Let $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Show that $\mathbb{R}$ together with the induced topology (from the standard topology on the target) is not homeomorphic to $\mathbb{R}$ with the standard topology.

Exercise 9. Let M be a subset of a metric space N. Let N have the metric topology. Then there is a metric on M such that the subspace topology is the metric topology on M.

Lemma 5.4. *Let Y be a topological space, let $X \subseteq Y$ be a subspace of Y, and $S \subseteq X$ be a subset of X. Write $\overline{S}^X$ for the closure of S in X, and $\overline{S}^Y$ for the closure of S in Y. Then*

$$\overline{S}^X = \overline{S}^Y \cap X.$$

Proof. Since $\overline{S}^Y \cap X$ is closed in X and contains S, we have $\overline{S}^X \subseteq \overline{S}^Y \cap X$.

Conversely, because $\overline{S}^X$ is a closed subset of X, there is a closed subset B in Y such that $\overline{S}^X = B \cap X$. Then

$$S \subseteq \overline{S}^X = B \cap X \subseteq B,$$

so that $\overline{S}^Y \subseteq B$ as well. It follows that

$$\overline{S}^X = B \cap X \supseteq \overline{S}^Y \cap X,$$

as desired. □

5.3 Open and closed maps

It will be convenient to introduce more terminology.

Definition 5.5. A continuous map is *open* if images of open subsets are open. A continuous map is *closed* if images of closed subsets are closed.

Examples 5.6. Homeomorphisms are both open and closed. But there are other maps that are open and closed and that are not homeomorphisms. For instance, the map $\mathbb{R} \to \star$ is such a map. Here and in the following, the symbol $\star$ is used for a space with a single point.

Definition 5.7. An *embedding* is a continuous injection where the source has the induced topology.

Proposition 5.8. *An embedding induces a homeomorphism onto the subspace defined by its image.*

Proof. By the properties of the induced topologies, the obvious bijection $x \mapsto f(x)$ is continuous, and so is its inverse. $\square$

Exercise 10. An embedding is open (or closed) if and only if the image is open (or closed).

6 Quotients

This section starts by turning around the directions of the arrows.

6.1 Co-induced topologies

Proposition 6.1. *Let X be a topological space, and let $f\colon X \to Y$ be a map from X into a set Y. There is a unique topology on Y such that a given map $t\colon Y \to T$ is continuous if and only the composition $tf\colon X \to T$ is continuous.*

Proof. First, we check that the set

$$\{\, V \subseteq Y \,|\, f^{-1}V \text{ open in } X \,\}$$

is a topology on Y.

Then, we check that it has the desired property: It makes f continuous, and if tf is continuous, and $W \subseteq T$ is open, then $(tf)^{-1}W = f^{-1}t^{-1}W$ is open in X, so that $t^{-1}W$ is open in Y, and t is continuous.

This shows that there exists a topology with the desired property. For uniqueness, proceed as in the proof in the preceding section. □

Definition 6.2. This topology is called the topology *co-induced by f on Y*.

6.2 Quotient spaces

The most important special case of the preceding construction occurs when f is surjective, for example if Y is the set of equivalence classes of an equivalence relation on X. (Note: Up to bijection, every surjection has this form.) In this case, the co-induced topology is commonly referred to as the *quotient topology*, and Y is a *quotient space* of X.

Exercise 11. Consider the quotient space Y of $\mathbb{R}$ with respect to the two equivalence classes $\{\, x \in \mathbb{R} \,|\, x < 0 \,\}$ and $\{\, x \in \mathbb{R} \,|\, x \geqslant 0 \,\}$. What is the topology on Y?

Example 6.3. We can built a circle from the interval $[\,0,1\,]$ by passing to the quotient space $[\,0,1\,]/\!\sim$, where we set $s \sim t$ if and only if $|t-s| \in \{\,0,1\,\}$. Continuous functions on this circle are in bijection with continuous functions f on $[0,1]$ such that $f(0) = f(1)$.

Example 6.4. For $n \geqslant 0$ let $\mathbb{R}\mathrm{P}^n$ be the set of 1-dimensional linear subspaces of the real vector space $\mathbb{R}^{n+1}$. There is a surjection

$$\mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{R}\mathrm{P}^n$$

that sends a point $x \neq 0$ to the line $\mathbb{R}x$ that passes through that point (and 0). The set $\mathbb{R}\mathrm{P}^n$ together with the quotient topology is n-dimensional real projective space.

Exercise 12. Show that there is a closed embedding $\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ whenever we have $m \leqslant n$.

6.3 Identifications

Just as embeddings are the invariant way to speak about subspaces, here is the invariant way to speak about quotient spaces.

Definition 6.5. An *identification* is a continuous surjection where the target has the co-induced topology.

Proposition 6.6. *A continuous surjection is an identification if it is open (or closed).*

Proof. Let $f\colon X \to Y$ be the map in question. We want to show that Y has the co-induced topology. So let $V \subseteq Y$. If it is open, by continuity, the pre-image $f^{-1}V$ is open. Conversely, if $f^{-1}V$ is open, so is

$$V = ff^{-1}V$$

by assumption on f. □

Example 6.7. The converse is not true: There are identifications where images of open subsets are not open. For example, consider again the quotient space Y of $\mathbb{R}$ with respect to the two equivalence classes $\{x \in \mathbb{R} \,|\, x \leqslant 0\}$ and $\{x \in \mathbb{R} \,|\, x > 0\}$. Look at the images of $]-2,-1[$ and $[1,2]$.

7 Products

For $j = 1,2$ let $f_j \colon X \to Y_j$ be two maps from a set X to two topological spaces Y_1 and Y_2.

Proposition 7.1. *There is a unique topology on X such that a given map $t \colon T \to X$ is continuous if and only both compositions $f_j \circ t \colon T \to Y_j$ are continuous.*

A proof follows almost the same pattern as the proofs of the corresponding results before. Only the start need a small extra idea: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are the induced topologies on X with respect to f_1 and f_2, then we need to have $\mathcal{T}_1 \cup \mathcal{T}_2$ contained in our topology. This need not be a topology on X, but it generates a topology $\langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$. This is the topology we want.

Remark 7.2. Since $\mathcal{T}_1$ is a topology, the union $\mathcal{T}_1 \cup \mathcal{T}_2$ contains X and is therefore certainly a subbasis for $\langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$. On can check that

$$\{ f_1^{-1} V_1 \cap f_2^{-1} V_2 \,|\, V_j \text{ open in } Y_j \}$$

is a basis for the topology $\langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$: Finite intersection of such subsets of X again have this form. The open subsets of X are unions of subsets of the form $f_1^{-1} V_1 \cap f_2^{-1} V_2$.

Exercise 13. Write out a full proof of the preceding proposition.

We are most interested in the case when $X = Y_1 \times Y_2$, and the $f_j = \mathrm{pr}_j$ are the projections to the factors: $\mathrm{pr}_j(y_1, y_2) = y_j$. Then the induced topology (as in the proposition above) is the *product topology* on $Y_1 \times Y_2$. It has the property that a map $t = (t_1, t_2) \colon T \to Y_1 \times Y_2$ is continuous if and only if its two factors (or components) $t_j = \mathrm{pr}_j t$ are continuous. A basis for the product topology is given by the subsets

$$V_1 \times V_2 = (V_1 \times Y_2) \cap (Y_1 \times V_2) = \mathrm{pr}_1^{-1} V_1 \cap \mathrm{pr}_2^{-1} V_2,$$

where the $V_j \subseteq Y_j$ are open.

Exercise 14. Let $Y_j = \{1,2\}$ with topology $\{\emptyset, \{j\}, Y_j\}$. Describe all open subsets of $Y_1 \times Y_2$ in the product topology.

Exercise 15. Show that there is a homeomorphism $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$, when $\mathbb{R}$ and $\mathbb{R}^2$ have the metric topology and $\mathbb{R} \times \mathbb{R}$ has the product topology.

Exercise 16. Show that the multiplication

$$\mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,y) \mapsto x \cdot y$$

is continuous. Hint: Since this is about a map *out of* a product, the property of the product topology is not helpful.

Exercise 17. Let M be a metric space with metric

$$d\colon M \times M \longrightarrow \mathbb{R}.$$

Show that d is continuous, when M has the metric topology and $M \times M$ has the product topology. Of course, the topology on $\mathbb{R}$ is the usual one.

Exercise 18. Show that the projections $\mathrm{pr}_j\colon Y_1 \times Y_2 \to Y_j$ are open. It follows that they are identifications. Are they closed as well?

Exercise 19. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Then the product topology on $A \times B$ is the same as the subspace topology (as a subset of $X \times Y$).

Exercise 20. Show that for all q in Y the subspace $X \times \{q\}$ of the product $X \times Y$ is homeomorphic to X.

Exercise 21. Let $f\colon X \to Y$ be a continuous map, and q a point in the target Y. Show that the subspace

$$\{\,(x,y) \in X \times Y \mid f(x) = y \text{ and } y = q\,\}$$

of $X \times Y$ is homeomorphic to the subspace

$$\{\,x \in X \mid f(x) = q\,\}$$

of X. Does this generalize the previous exercise?

8 Sums

Sums are obtained from products by turning the arrows around. Actually, since we have only discussed products $Y_1 \times Y_2$ of pairs (Y_1, Y_2) of spaces before, we would end up with sums $Y_1 + Y_2$ of pairs (Y_1, Y_2). In turns out that it costs not much more efforts to do sums of arbitrary families right away.

8.1 The disjoint union as a set

Let J be a set, and let $(X_j \,|\, j \in J)$ be a family of sets X_j indexed by the set J. We form the union

$$X = \bigcup_{j \in J} X_j \tag{8.1}$$

as a set. Then we can consider the subset

$$S = \{\, (x, j) \in X \times J \,|\, x \in X_j \,\}$$

of that. This has subsets

$$S_j = S \cap (X \times \{j\}) = X_j \times \{j\}.$$

There are canonical bijections $S_j \cong X_j$. Note that

$$S = \bigcup_{j \in J} S_j$$

and that

$$S_j \cap S_k = \varnothing$$

if $j \neq k$.

Definition 8.1. The set S is the *disjoint union* or *sum* of the family $(X_j \,|\, j \in J)$. It is written

$$\coprod_{j \in J} X_j \text{ or } \sum_{j \in J} X_j.$$

Remark 8.2. If $\mathcal{X}$ is a set of sets, then we can take the family $(X \,|\, X \in \mathcal{X})$ of sets and proceed as above. Conversely, a family $(X_j \,|\, j \in J)$ of sets determines a set $\{X_j \,|\, j \in J\}$ of sets. In order to be able to 'multiply' a given set, it is preferable to work with families here.

8.2 A topology on the disjoint union

Let J be a set, and let $(X_j \mid j \in J)$ be a family of topological spaces X_j with topologies $\mathcal{T}_j$.

There is a unique topology $\mathcal{T}'_j$ on the sets S_j (as above) such that the canonical bijections $S_j \cong X_j$ are homeomorphisms for each j in J: Choose a bijection, and take the (co-)induced topology, depending on the direction of the bijection chosen.

Set

$$\mathcal{T} = \{U \subseteq S \mid U \cap S_j \in \mathcal{T}'_j \text{ for all } j \in J\}. \tag{8.2}$$

This is a topology on S.

Definition 8.3. The topological space S (with the topology $\mathcal{T}$) is called the topological *sum* of the family $(X_j \mid j \in J)$, and is written as

$$\coprod_{j \in J} X_j \text{ or } \sum_{j \in J} X_j.$$

Proposition 8.4. *The injections*

$$\mathrm{in}_k \colon X_k \cong S_k \subseteq S = \coprod_{j \in J} X_j$$

are open and closed embeddings for all $k \in J$.

Proof. The definition immediately implies that $S_k \subseteq S$ has the subspace topology, so that in_k is an embedding. Therefore, it it will be open if the image S_k is open in the sum. But the intersection with S_j is either empty (if $j \neq k$) or S_j (if $j = k$), hence open. Finally, the subset S_k is also closed, because its complement it the union of the S_j for $j \neq k$, which is open by what we have just seen. □

Proposition 8.5. *The map that sends a continuous map*

$$t \colon \coprod_{j \in J} X_j \longrightarrow T$$

to the family $(t\,\mathrm{in}_j \mid j \in J)$ of continuous maps $X_j \to T$ is a bijection.

In this sense, continuous maps out of a sum are 'the same' as families of continuous maps out of the summands.

Proof. Given a family $(f_j \mid j \in J)$ of continuous maps $X_j \to T$, there is clearly a unique map

$$t\colon \coprod_{j\in J} X_j \longrightarrow T$$

of sets with $t\,\mathrm{in}_j = f_j$ for all $j \in J$: Set $t(x,j) = f_j(x)$. By definition of the topology, this map is continuous. This construction gives an inverse to the map $t \mapsto (t\,\mathrm{in}_j \mid j \in J)$. $\square$

8.3 An application

Let X be a topological space and let $(U_j \mid j \in J)$ be a family of open subsets of X such that

$$X = \bigcup_{j\in J} U_j.$$

Example 8.6. A good example to keep in mind is a metric space X together with the set of all balls inside X. Or any topological space with a subbasis for its topology.

Lemma 8.7. *In the situation above, a subset V of X is open in X if and only if the intersection $V \cap U_j$ is open in U_j for all $j \in J$.*

Proof. One direction is clear from the definition of the subspace topology. For the other one, the equality

$$V = \bigcup_{j\in J} V \cap U_j$$

can be used. $\square$

Let each U_j have the subspace topology, and form their sum

$$\coprod_{j\in J} U_j$$

as above. Note that $(u,j) \mapsto u$ defines a surjective map

$$f\colon \coprod_{j\in J} U_j \longrightarrow X.$$

Proposition 8.8. *The map f is an open surjection.*

Proof. The image of an open subset V of the sum is essentially

$$\bigcup_{j\in J} V\cap U_j,$$

and therefore open. $\square$

It follows that the map f is an identification, so that X can be thought of (up to homeomorphism) as a quotient space of the sum of the family $(U_j \mid j\in J)$.

Corollary 8.9. *In the given situation, a map $f\colon X\to Y$ is continuous if and only if all of its restrictions $f|U_j\colon U_j\to Y$ are continuous.*

Exercise 22. Is this section, is it necessary that the subsets $U_j\subseteq X$ are open? Hint: If U is any topological space, we get a continuous bijection

$$\coprod_{u\in U}\{u\}\longrightarrow U.$$

Chapter III

Topological properties

9 Connected spaces

How do we recognize if a topological space X is a topological sum in a non-trivial way? With symbols: When is

$$X \cong X_1 + X_2$$

with $X_1 \neq \emptyset \neq X_2$?

Proposition 9.1. *A topological space X is a topological sum in a non-trivial way if and only if there is a subset X_1 of X that is both open and closed in X and different from $\emptyset$ and X.*

Proof. Use that (a copy of) X_1 is open and closed in $X_1 + X_2$. Conversely, if X_1 is such a subset, then $U \subseteq X$ is open if and only if $U \cap X_1$ and $U \cap (X \setminus X_1)$ are open in X_1 and $X \setminus X_1$, so that $X \cong X_1 + X_2$ with $X_2 = X \setminus X_1$. □

Definition 9.2. A topological space X is called *connected* if $\emptyset$ and X are the only subsets of X that are both open and closed.

Example 9.3. For all points p in $\mathbb{R}$, the subspace $\mathbb{R} \setminus p$ is not connected. In fact, if C is a connected subspace of $\mathbb{R}$, and $a < b < c$ with a, c in C, then also b in C.

Exercise 23. Let X be a discrete space. When is X connected? Same with 'indiscrete' instead of 'discrete.'

Example 9.4. Subspaces of connected spaces need not be connected.

Proposition 9.5. *Let X be connected and $f\colon X \to Y$ be a continuous surjection. Then Y is connected. In particular, quotients of connected spaces are connected.*

Proposition 9.6. *Let $\mathcal{C}$ be a set of subspaces C of a topological space X such that*

$$X = \bigcup_{C \in \mathcal{C}} C$$

and

$$C_1 \cap C_2 \neq \emptyset$$

for all $C_1, C_2 \in \mathcal{C}$. If all subspaces C in $\mathcal{C}$ are connected, so is X.

Proof. Let $P \neq \emptyset$ be a subset of X that is open and closed. It suffices to show that $P = X$. Since X is the union of the C, there has to be a C_0 such that $P \cap C_0 \neq \emptyset$. Since C_0 is connected, we have $P \cap C_0 = C_0$, or $C_0 \subseteq P$. Given any other C in $\mathcal{C}$, we have

$$P \cap C \supseteq C_0 \cap C \neq \emptyset,$$

and we get $C \subseteq P$ as before. Since X is the union of the C, it follows that $X \subseteq P$, and the converse inclusion is clear. □

Proposition 9.7. *If X and Y are connected spaces, so is $X \times Y$.*

Proof. We can assume that $X \neq \emptyset \neq Y$. Then we can pick a point (x_0, y_0) in $X \times Y$. If (x, y) is any point in $X \times Y$, we know that $X \times \{y_0\}$ and $\{x\} \times Y$ are connected. Their intersection, the singleton $\{(x, y_0)\}$, is connected, and so is their union

$$C(x,y) = (X \times \{y_0\}) \cup (\{x\} \times Y)$$

by Proposition 9.6. The points (x_0, y_0) and (x, y) both lie in $C(x, y)$, so that we can apply Proposition 9.6 again to the set of all $C(x, y)$ in order to deduce that $X \times Y$ is connected. □

Example 9.8. Obviously, sums of connected spaces need not be connected.

Exercise 24. Let C be a connected subspace of a topological space Y. Then all subspaces X of Y such that

$$C \subseteq X \subseteq \overline{C}$$

are also connected. In particular, the closure $\overline{C}$ of C in Y is connected.

The following result may explain why the proof of Proposition 9.7 may seem unexpectedly complicated: Whereas product are made to map into, connectivity (as defined here) is related to maps out of the space in question.

Proposition 9.9. *A topological space X is connected if and only if all continuous maps from X to discrete spaces are constant.*

Proof. A typical disconnected space X has the form $X_1 + X_2$ with X_j non-empty. Then the map $X \to \{1, 2\}$ that sends X_j to j is continuous and not constant.

Conversely, let $f\colon X \to Y$ be a continuous map into a discrete space that is not constant. Pick a value y. Then $f^{-1}\{y\} \subseteq X$ is open and closed, and neither $\emptyset$ nor X. □

We can now turn our attention towards a notion of connectivity that is made out of maps into the space in question.

9.1 Path connectivity

The following result is one form of the *intermediate value theorem* from calculus.

Theorem 9.10. *The space $\mathbb{R}$ (with its usual topology) is connected.*

Proof. Let $P \subseteq \mathbb{R}$ be open, closed, with $a \in P$ and $b \notin P$. We can replace P by its complement $\mathbb{R} \setminus P$ if necessary to make sure that $a < b$. The subset $P \cap [a,b]$ is non-empty ($a \in P$) and bounded (by b). It follows that there is a supremum s. We will lead this to a contradiction. Since P is open, the point a cannot be the supremum. Since $\mathbb{R} \setminus P$ is open, the point b cannot be the supremum. It follows that $a < s < b$. We cannot have $s \in P$, again because P is open. And we cannot have $s \in \mathbb{R} \setminus P$, again because $\mathbb{R} \setminus P$ is open. This leaves no possibilities. □

It follows that all intervals $]a,b[$, $[a,b]$, $[a,b[$, and $]a,b]$ for $a < b$ are connected.

Theorem 9.11. *Let $f\colon [0,1] \to [0,1]$ be a continuous self-map. Then f has a fixed point: There is a t such that $f(t) = t$.*

Proof. A fixed point of f is a zero of $g\colon [0,1] \to \mathbb{R}$ with $g(t) = f(t) - t$. If we have $f(0) = 0$ or $f(1) = 1$ we are done. Otherwise we have $g(0) > 0$ and $g(1) < 0$. Since $[0,1]$ is connected, so is the image $g[0,1]$. Hence 0 is in the image, so that f has a fixed point. □

Definition 9.12. A topological space X is called *path-connected* if (it is non-empty) and for each pair (x_0, x_1) of points there is a continuous map $f\colon [0,1] \to X$ such that $f(t) = x_t$ for $t = 0$ and $t = 1$.

Proposition 9.13. *Path-connected spaces are connected.*

Proof. Let P be a subset of X that is open and closed. If we can pick $x_0 \in P$ and $x_1 \in X \setminus P$, and a continuous map $f\colon [0,1] \to X$ such that $f(t) = x_t$ for $t = 0$ and $t = 1$, then $f^{-1}P$ would be a subset of $[0,1]$ that is open and closed, with $0 \in P$, but $1 \notin P$. This contradicts the connectivity of $[0,1]$. □

Examples 9.14. For all $n \geqslant 2$ that space $\mathbb{R}^n$ is connected, and so is $\mathbb{R}^n \setminus \star$. It follows that $\mathbb{R} \ncong \mathbb{R}^n$ for any $n \geqslant 2$.

10 Hausdorff spaces

Definition 10.1. A topological space X a *Kolmogorov space* if for any two different points $p \neq q$ in X there exists an open subset U of X that contains precisely one of them (and not the other).

Exercise 25. Let X be a topological space with the property that every continuous self-map $f \colon X \to X$ has a fixed point. (This means that there is a point x, that may depend on f, such that $f(x) = x$.) Show that X is a Kolmogorov space.

Proposition 10.2. *For a topological space X the following are equivalent:*
(1) *For any two different points $p \neq q$ in X there exists open subsets $U = U(p,q)$ and $V = V(p,q)$ such that $p \in U \setminus V$ and $q \in V \setminus U$.*
(2) *For every point x in X the subset $\{x\}$ is closed in X.*

Proof. If points are closed in a space X, we can always take $U(p,q) = X \setminus q$ and $V(p,q) = X \setminus p$. Conversely, given any point p in a space that meets condition (1), we have

$$X \setminus p = \bigcup_{q \neq p} V(p,q),$$

and this is open. □

Example 10.3. The topological space $\Omega = \{\alpha, \omega\}$ with topology $\{\emptyset, \{\omega\}, \Omega\}$ is a Kolmogorov space where not all points are closed.

Definition 10.4. A topological space X a *Hausdorff space* if or any two different points $p \neq q$ in X there are disjoint open subsets U and V such that $p \in U \setminus V$ and $q \in V \setminus U$.

Examples 10.5. Clearly, in Hausdorff spaces all points are closed, but the converse does not hold: take the co-finite topology on an infinite set.

Examples 10.6. Every metric space is a Hausdorff spaces.

Exercise 26. Every subspace of a Hausdorff space is a Hausdorff space.

Exercise 27. Every product of (two) Hausdorff spaces is a Hausdorff space.

Exercise 28. Every sum of Hausdorff spaces is a Hausdorff space.

Proposition 10.7. *A topological space X is a Hausdorff space if and only if the diagonal*

$$\Delta_X = (\mathrm{id}_X, \mathrm{id}_X)\colon X \to X \times X, x \mapsto (x,x)$$

is a closed embedding.

Proof. Use: The diagonal is closed if and only if its complement is open. And: An open rectangle $U \times V$ lies in the complement of the diagonal if and only if U and V are disjoint. □

Corollary 10.8. *Let $f, g\colon X \to Y$ be continuous maps into a Hausdorff space Y. Then the subset*

$$\{\, x \in X \mid f(x) = g(x) \,\}$$

of X is closed.

Proof. This is the pre-image of the closed diagonal $\Delta(Y)$ under the continuous map $(g, f)\colon X \to Y \times Y$. □

Corollary 10.9. *Let $f\colon X \to Y$ be a continuous maps into a Hausdorff space Y. Then its graph*

$$\{\, (x,y) \in X \times Y \mid f(x) = y \,\}$$

is closed in $X \times Y$.

Proof. This is the pre-image of the closed diagonal $\Delta(Y)$ under the continuous map $f \times \mathrm{id}_Y\colon X \times Y \to Y \times Y$. □

10.1 Quotients of Hausdorff spaces

A quotient of a Hausdorff space need not be a Hausdorff space.

Example 10.10. Let $\mathbb{R}$ be the space of real numbers with the usual metric topology. Here is an equivalence relation on it:

$$v \sim w \Leftrightarrow uv = w \text{ for some } u \in \mathbb{R}^{\times}.$$

There are two equivalence classes $[0]$ and $[1]$. The quotient topology on the set of these two elements is such that $\{\,[1]\,\}$ is open, whereas $\{\,[0]\,\}$ is not. This is clearly not a Hausdorff space.

An equivalence relation $\sim$ on X is given by the set

$$R(\sim) = \{ (p,q) \in X \times X \,|\, p \sim q \}$$

of equivalent pairs of points. More generally, given any map $f\colon X \to Y$, we can consider

$$R(f) = \{ (p,q) \in X \times X \,|\, f(p) = f(q) \}.$$

Proposition 10.11. *Let $f\colon X \to Y$ be continuous and Y be Hausdorff. Then the subset $R(f)$ is closed in $X \times X$.*

Proof. The subset $R(f)$ is the pre-image of the diagonal under the continuous map $f \times f$. □

Proposition 10.12. *Let $f\colon X \to Y$ be an open surjection. If $R(f)$ is a closed subset in $X \times X$, then Y is a Hausdorff space.*

Proof. Assume that $f(p) \neq f(q)$. Then (p,q) is not in $R(f)$, and we can find an open rectangle $U \times V$ in $X \times X$ that does not meet $R(f)$. Then the subsets $f(U)$ and $f(V)$ of Y are, by assumption, disjoint and open, and we have $p \in f(U)$ as well as $q \in f(V)$. □

10.2 Separated maps

Definition 10.13. A continuous map $f\colon X \to Y$ is called *separated* if any two points $x_1 \neq x_2$ in X with $f(x_1) = f(x_2)$ can be separated in the sense that here exists $U_j \subseteq X$ open such that $x_j \in U_j$ for $j = 1,2$ and $U_1 \cap U_2 = \emptyset$.

Exercise 29. The unique map $X \to \star$ to a singleton space $\star$ is separated if and only if X is a Hausdorff space.

Exercise 30. Let X be any topological space. Is the identity map $\mathrm{id}_X\colon X \to X$ separated? Are compositions of separated maps separated?

Exercise 31. Let X be a Hausdorff space. Show that all maps $X \to Y$ are separated. Is the converse also true?

Exercise 32. Let $f\colon X \to Y$ be separated. Show that the *fibers* $f^{-1}\{y\} \subseteq X$ for all $y \in Y$ are Hausdorff spaces. Is the converse also true?

10.3 Fiber products (pullbacks)

Definition 10.14. Let $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ be two continuous maps with the same target. The subspace

$$X_1 \times_Y X_2 = \{\,(x_1,x_2) \in X_1 \times X_2 \,|\, f_1(x_1) = f_2(x_2)\,\}$$

of the product is called the *fiber product*. (The maps f_1 and f_2 are not in the notation.)

The fiber product comes with two continuous (!) maps

$$X_1 \xleftarrow{p_1} X_1 \times_Y X_2 \xrightarrow{p_2} X_2,$$

with $p_j(x_1,x_2) = x_j$.

Exercise 33. Write $R(\sim)$ and $R(f)$ above as fiber products.

Exercise 34. If f_2 is separated, show that p_1 is separated.

Exercise 35. Show that $f\colon X \to Y$ is separated if and only if the image of the diagonal map

$$\Delta\colon X \longrightarrow X \times_Y X$$

is a closed subset of the fiber product. Is this the case if and only if the diagonal map Δ is closed (as a continuous map)?

Exercise 36. Let $f\colon X \to Y$ be a separated map, and let $s\colon Y \to X$ be a continuous map such that $fs = \mathrm{id}_Y$. Show that the image $s(Y) \subseteq X$ of the map s is a closed of X. Is this the case if and only if s is closed (as a continuous map)?

11 Compact spaces

Definition 11.1. Let X be a topological space. A set $\mathcal{S}$ of subsets $S \subseteq X$ of X is a *cover* if

$$X = \bigcup_{S \in \mathcal{S}} S.$$

A cover is called *open* if all S are open in X. A cover is called *finite* if $\mathcal{S}$ is a finite set. If two covers satisfy $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}$ is called a *subcover* of $\mathcal{T}$.

Definition 11.2. A topological space is *compact* if every open subcover of it has a finite subcover.

Examples 11.3. All finite spaces are compact. An infinite space with the indiscrete or co-finite topology is compact.

Example 11.4. The space $\mathbb{R}$ with its usual topology is not compact. In fact, any compact metric space is a ball (or empty), and $\mathbb{R}$ is not.

11.1 Three propositions

Proposition 11.5. *Let X be a compact space, and $A \subseteq X$ be a closed subset. Then A is compact with respect to the subspace topology.*

Proof. Let $\mathcal{U}$ be a set of open subsets of X such that

$$A \subseteq \bigcup_{U \in \mathcal{U}} U.$$

Then the $U \cap A$ form an open cover of A, and every open cover has this form. Since $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of X, there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0 \cup \{X \setminus A\}$ covers X. Then the $U \cap A$ with U in $\mathcal{U}_0$ cover A. □

Proposition 11.6. *Let X be a compact space, and $f \colon X \to Y$ be a continuous map. Then the image $f(X)$ is a compact subspace of Y.*

Proof. Let $\mathcal{V}$ be a set of open subsets of Y such that

$$f(X) \subseteq \bigcup_{V \in \mathcal{V}} V.$$

Then the $V \cap f(X)$ form an open cover of $f(X)$, and every open cover has this form. The pre-images $f^{-1}V$ form an open cover of X, so that there is a finite subcover by the $f^{-1}V$ for some finite subset $\mathcal{V}_0 \subseteq \mathcal{V}$. Then the $V \cap f(X)$ for V in $\mathcal{V}_0$ form an open cover of $f(X)$, a subcover of the cover that we started with. □

Proposition 11.7. *Let X be a Hausdorff space, and $K \subseteq X$ be a compact subspace. Then K is closed in X.*

Proof. We show that the complement is open. Let x be in the complement. Then for each k in K we pick open $U(k)$ and $V(k)$ in X that are disjoint, with k in $U(k)$ and x in $V(k)$. Since K is compact,

$$K \subseteq \bigcup_{k \in K_0} U(k)$$

for a finite subset $K_0 \subseteq K$. Then

$$V = \bigcap_{k \in K_0} V(k)$$

is open, contains x, and does not meet K. □

11.2 Three corollaries

Corollary 11.8. *If $f\colon X \to Y$ is a continuous map from a compact space X to a Hausdorff space Y, then f is closed.*

Proof. This just puts the three preceding propositions together. □

Corollary 11.9. *If $f\colon X \to Y$ is a continuous bijection between a compact space X and a Hausdorff space Y, then f is a homeomorphism.*

Proof. This is a special case of the preceding corollary. □

Corollary 11.10. *If $f\colon X \to Y$ is a continuous surjection from a compact space X onto a Hausdorff space Y, then f is an identification.*

11.3 Proper maps

Proposition 11.11. *Every continuous map $f\colon X \to Y$ from a compact space X to a Hausdorff space Y has compact fibers.*

Proof. If y is in Y, then $\{y\}$ is closed, because Y is Hausdorff. It follows that the fibers are closed. Then they are compact, since X is compact. □

Definition 11.12. A closed map $f\colon X \to Y$ with compact fibers is called *proper*.

Corollary 11.13. *Every continuous map $f\colon X \to Y$ from a compact space X to a Hausdorff space Y is proper.*

Example 11.14. A topological space X is compact if and only if the map $X \to \star$ is proper.

Example 11.15. The identity map $\mathrm{id}_X\colon X \to X$ is proper for all topological spaces X.

Theorem 11.16. *Let $f\colon X \to Y$ be a continuous map. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*, where*
(1) *The map f is closed and has compact fibers (i.e. f is proper).*
(2) *Pre-images under f of compact subspaces of Y are compact in X.*
(3) *The map f has compact fibers.*

Proof. The implication (2) $\Rightarrow$ (3) is clear, and we only need to care about the direction (1) $\Rightarrow$ (2): Given $K \subseteq Y$ compact, we have to show that $f^{-1}K \subseteq X$ is compact.

Let $\mathfrak{U}$ be a set of open subsets of X such that $f^{-1}K$ is contained in the union of the elements of $\mathfrak{U}$. Since the fibers are compact: For all k in K there is a finite subset $\mathfrak{U}(k) \subseteq \mathfrak{U}$ such that the fiber $f^{-1}\{k\}$ is contained in the union of the elements of $\mathfrak{U}(k)$. Let us denote this union by $V(k)$:

$$V(k) = \bigcup_{U \in \mathfrak{U}(k)} U.$$

Since $V(k)$ is open in X, the complement $X \setminus V(k)$ is closed in X. By assumption, the image $f(X \setminus V(k))$ is closed in Y, so that its complement

$$W(k) = Y \setminus f(X \setminus V(k))$$

is open in Y. By definition, we have $k \in W(k)$ and $f^{-1}W(k) \subseteq V(k)$. From the first of these, it follows that these set yield an open cover of K. By hypothesis, there is a finite subset $K_0 \subseteq K$ such that

$$K \subseteq \bigcup_{k \in K_0} W(k).$$

Now we have

$$f^{-1}K \subseteq f^{-1} \bigcup_{k \in K_0} W(k) \subseteq \bigcup_{k \in K_0} f^{-1}W(k) \subseteq \bigcup_{k \in K_0} V(k) \subseteq \bigcup_{k \in K_0} \bigcup_{U \in \mathcal{U}(k)} U$$

so that

$$\mathcal{U}_0 = \bigcup_{k \in K_0} \mathcal{U}(k)$$

yields a finite subcover of $\mathcal{U}$. □

Corollary 11.17. *Compositions of proper maps are proper.*

Proof. Compositions of closed maps are closed. For the fibers, use $(1) \Rightarrow (2)$ from the theorem before. □

Example 11.18. Let X be a finite set with more than one point. Consider the identity from X with the discrete topology to X with the indiscrete topology. This map is not closed, but pre-images of compact subspaces are compact. This shows that (2) does not imply (1).

Example 11.19. Let X be a infinite set. Consider the identity from X with the discrete topology to X with the indiscrete topology. This map has compact fibers, but X is compact in the indiscrete topology, but not in the discrete topology. Therefore, not all pre-images of compact subspaces are compact. This shows that (3) does not imply (2).

11.4 Fiber products

Theorem 11.20. *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \underset{f}{\longrightarrow} & B \end{array}$$

be a diagram that displays X' as the fiber product of B' and X over B. If p is proper, then so is p'.

Proof. We have already seen (easily) that the fibers of p' are homeomorphic to the fibers of p. By assumption, those of p are compact. Therefore, so are those of p'. It remains to be checked that p' is closed.

Let $A' \subseteq X'$ be closed. We need to show that $p'(A') \subseteq B'$ is closed. To do so, let b' be a point in the complement $B' \setminus p'(A')$.

If x in X is any point in the fiber of p over $f(b')$, then (b',x) is in the fiber product X', but not in A': If it were in A', then $b' = p'(b',x)$ were in $p'(A')$, contrary to our assumption on b'.

Since A' is closed, we can then find open subsets $U(x) \subseteq B'$ and $V(x) \subseteq X$ such that $b' \in U(x)$, $x \in V(x)$, and

$$(U(x) \times V(x)) \cap A' = \emptyset.$$

Since the fiber of p over $f(b')$ is compact, there is a finite set F of points x in the fiber such that the $V(x)$ with x in F cover the fiber. The open set

$$V = \bigcup_{x \in F} V(x)$$

contains that fiber, and the open set

$$U = \bigcap_{x \in F} U(x)$$

contains b', and

$$(U \times V) \cap A' = \emptyset.$$

Since p is closed, the subset

$$W = B \setminus p(X \setminus V)$$

is open, and it contains $f(b')$: The fiber over $f(b')$ is contained in V, so $f(b')$ is not in $p(X \setminus V)$.

By continuity of f there is an open subset

$$U' \subseteq U$$

such that $b' \in U'$ and $f(U') \subseteq W$.

We are done if we can show that $U' \cap p'(A') = \emptyset$. But, if that were not the case, then there were a $u' \in U'$ and an $x \in X$ such that $(u', x) \in A'$. Then $f(u') \in W$ and $p(x) = f(u')$, so that $p(x) \in W$. This means $x \notin X \setminus V$, or $x \in V$. Then

$$(u', x) \in (U' \times V) \cap A' \subset (U \times V) \cap A' = \emptyset,$$

a contradiction. □

Corollary 11.21. *Products of proper maps are proper.*

Proof. If $g\colon X \to Y$ and $g'\colon X' \to Y'$ are proper, then

$$g \times g'\colon X \times X' \longrightarrow Y \times Y'$$

is the composition $(g \times \mathrm{id}_{X'})(\mathrm{id}_Y \times g')$. By what has been shown above, we can therefore assume that one of the maps is an identity. This case follows from the fact that

$$\begin{array}{ccc} X \times X' & \longrightarrow & X \\ {\scriptstyle g \times \mathrm{id}_{X'}} \downarrow & & \downarrow {\scriptstyle g} \\ Y \times X' & \xrightarrow[\mathrm{pr}_1]{} & Y \end{array}$$

identifies $g \times \mathrm{id}_{X'}$ with g' (with respect to $f = \mathrm{pr}_1$), and similarly for the other factor. □

Corollary 11.22. *Products of compact spaces are compact.*

11.5 Compact subspaces of the real line

In this section we will recognize the compact subspaces of $\mathbb{R}^n$. We will begin by an example.

Example 11.23. The subspace $[0,1]$ of $\mathbb{R}$ is compact (with respect to the standard metric topology). To see this, let $\mathcal{U}$ be an open cover. We consider the subset X of $[0,1]$ that consists of the numbers x such that $[0,x]$ is contained in a finite subcover. We would like to show that 1 is one of them.

Clearly, the set X is not empty, since 0 is in X. Also, the set X is bounded, by definition. Then the set X has a supremum s, say. Since s is in one of the open subsets U of $\mathcal{U}$, and every such U contains an element of X, we see that s is in X.

In fact, this argument shows that also some elements of $[0,1]$ that are larger than s would lie in X, if $s < 1$. Because that cannot happen, we deduce $s = 1$, and we are done.

Theorem 11.24. *A subspace $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. On direction is more or less obvious: Compact subspaces of Hausdorff spaces are closed, and compact subspaces of metric spaces are bounded.

Conversely, if X is bounded, then X sits inside a cube $[a,b]^n$. By the example above, and a previous theorem. This cube is compact. If X is also closed in $\mathbb{R}^n$, it is closed in $[a,b]^n$. (The inclusion $[a,b]^n \subseteq \mathbb{R}^n$ is a closed embedding.) Now X, as a closed subspace of a compact space, is itself compact. □

Chapter IV

Nets and filters

12 From sequences over nets to filters

This section is mostly about language concerning sets. In the next section we will use it to rephrase some of the main topological notions that we have seen so far.

12.1 Sequences

Let X be a set.

Definition 12.1. A *sequence* in X is a map $\mathbb{N} \to X$. This is usually written (x_n) to indicate that the natural number n is sent to x_n.

Definition 12.2. Let S be a subset of X. We say that a sequence (x_n) in X *eventually lies in* S if there is a $n_S \in \mathbb{N}$ such that $x_n \in S$ for all $n \geqslant n_S$.

Definition 12.3. We say that a sequence (x_n) *converges* to x_∞ and that x_∞ is a *limit* of (x_n) if the sequence lies eventually in every open subset of X that contains x_∞.

There are various reasons why we would like to generalize this to more general indexing sets than $\mathbb{N}$. For example, we won't have to re-index subsequences anymore, there are no size limitation due to the countability of $\mathbb{N}$, and last but not least: This is *General* Topology!

12.2 Directed posets

We need to abstract from the properties of $\mathbb{N}$ used in the defintions above.

Definition 12.4. A *partial order* on a set Λ is a (binary) relation $\leqslant$ such that the axioms
(1) $\lambda \leqslant \lambda$
(2) $\lambda \leqslant \mu$ and $\mu \leqslant \nu$ implies $\lambda \leqslant \nu$
(3) $\lambda \leqslant \mu$ and $\mu \leqslant \lambda$ implies $\lambda = \mu$
hold.

If $\leqslant$ is a partial order on Λ, then $(\Lambda, \leqslant)$ is called a partially ordered set. Of course, we will often just write Λ for it.

Definition 12.5. A *directed set* is a partially ordered set $(\Lambda, \leqslant)$ that is non-empty and such that for all paris (λ_1, λ_2) in Λ there is a μ in Λ such that $\lambda_1 \leqslant \mu$ and $\lambda_2 \leqslant \mu$.

Examples 12.6. The set $\mathbb{N}$ is directed (with respect to the usual order), as is $\mathbb{Z}$, or

$$\{b \in \mathbb{Z} \,|\, a \leqslant b\}$$

for each $a \in \mathbb{Z}$. The set of even integers is directed, and the set of prime numbers is directed.

These examples all have the property that $\lambda \leqslant \mu$ or $\mu \leqslant \lambda$ for all λ and μ. But this does not have to be the case:

Example 12.7. Let $\Lambda = \mathbb{N} \times \mathbb{N}$. We set $(a_1, a_2) \leqslant (b_1, b_2)$ if $a_j \leqslant b_j$ for both $j = 1, 2$. In that case $(1,2) \nleqslant (2,1)$ and $(2,1) \nleqslant (1,2)$. But given $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$, we have $\alpha, \beta \leqslant \mu$ for

$$\mu = (\max\{a_1, b_1\}, \max\{a_2, b_2\}).$$

It follows that $\Lambda = \mathbb{N} \times \mathbb{N}$ is a directed set.

This example is clearly relevant when studying ‘double’ sequences.

12.3 Nets

Let X be a set.

Definition 12.8. A *net* (or a *Moore–Smith sequence*) in X is a map $\Lambda \to X$ that is defined on some directed set Λ. This is usally written (x_λ) to indicate that $\lambda \in \Lambda$ is sent to $x_\lambda \in X$.

Example 12.9. Every sequence (x_n) is a net; take $\Lambda = \mathbb{N}$.

Definition 12.10. Let S be a subset of X. A net (x_λ) in X lies *eventually in S* if there is a $\lambda_S \in \Lambda$ such that $x_\lambda \in S$ for all $\lambda \geqslant \lambda_S$.

For sequences, this is the same as before.

Definition 12.11. We say that a net (x_λ) *converges* to x_∞ and that x_∞ is a *limit* of (x_λ) if the net lies eventually in every open subset of X that contains x_∞.

For sequences, this is the same as before.

12.4 Filters

Let (x_λ) be a net in X. Consider the set

$$\mathcal{F}(x_\lambda) = \{ S \subseteq X \,|\, (x_\lambda) \text{ is eventually in } S \}$$

of subsets of X.

Proposition 12.12. *The properties*
(1) $\emptyset \notin \mathcal{F}(x_\lambda)$
(2) $X \in \mathcal{F}(x_\lambda)$
(3) $S \in \mathcal{F}(x_\lambda)$ *and* $S \subseteq T \subseteq X$ *imply* $T \in \mathcal{F}(x_\lambda)$
(4) $S_1, S_2 \in \mathcal{F}(x_\lambda)$ *imply* $S_1 \cap S_2 \in \mathcal{F}(x_\lambda)$
hold for the set $\mathcal{F}(x_\lambda)$.

Proof. For (1) use that Λ is not empty. For (4) use that Λ is directed. □

This can be turned into a definition.

Definition 12.13. Let X be a set. A set $\mathcal{F}$ of subsets with
(1) $\emptyset \notin \mathcal{F}$
(2) $X \in \mathcal{F}$
(3) $S \in \mathcal{F}$ and $S \subseteq T \subseteq X$ imply $T \in \mathcal{F}$
(4) $S_1, S_2 \in \mathcal{F}$ imply $S_1 \cap S_2 \in \mathcal{F}$
is called a *filter* on X. If $S \in \mathcal{F}$, then we will say that the filter $\mathcal{F}$ *eventually lies* in the set S.

Proposition 12.12 can be rephrased to say that $\mathcal{F}(x_\lambda)$ is a filter on X whenever (x_λ) is a net in X.

Examples 12.14. If x is an element in a set X then

$$\mathcal{P}(x) = \{ S \subseteq X \,|\, x \in S \}$$

is the *principal filter* defined by x.

Exercise 37. Show that $\{x\}$ is contained in a filter $\mathcal{F}$ if and only if $\mathcal{F} = \mathcal{P}(x)$.

Examples 12.15. If X happens to be a topological space, then

$$\mathcal{O}(x) = \{ S \subseteq X \,|\, U \subseteq S \text{ for some } U \in \mathcal{P}(x) \text{ that is open in } X \}$$

is the *neighborhood filter* defined by x. In other words: We say that a subset S of a space X is in the neighborhood filter of a point x (or shorter: a *neighborhood* of x in X) if there exists an open subset U of the space X that contains x and that is contained in S. So U has to be open, while S typically won't be.

Proposition 12.16. *Let (x_λ) be a net in a topological space X. Then x_∞ is a limit of the net (x_λ) if and only if $\mathcal{O}(x_\infty) \subseteq \mathcal{F}(x_\lambda)$.*

The generalization to filters is now obvious:

Definition 12.17. Let $\mathcal{F}$ be a filter on a topological space X. We say that x is a *limit* of the filter $\mathcal{F}$ if the filter contains the neighborhood filter: $\mathcal{O}(x) \subseteq \mathcal{F}$.

12.5 Bases

Let $\mathcal{F}$ be a filter on a set X.

Definition 12.18. A subset $\mathcal{B} \subseteq \mathcal{F}$ is called a *basis* for $\mathcal{F}$ if all $F \in \mathcal{F}$ contain a $B \in \mathcal{B}$.

Proposition 12.19. *Let $\mathcal{B}$ be a non-empty set of non-empty subsets of X such that for all $B_1, B_2 \in \mathcal{B}$ there is a $B \in \mathcal{B}$ such that $B \subseteq B_1 \cap B_2$. Then*

$$\mathcal{F} = \{\, S \subseteq X \,|\, S \text{ contains a } B \in \mathcal{B} \,\}$$

is a filter on X with basis $\mathcal{B}$.

Exercise 38. Prove this.

12.6 Ultrafilters

Definition 12.20. A maximal filter $\mathcal{F}$ on a set X is called an *ultrafilter*.

Here 'maximal' means that for each filter $\mathcal{G}$ on X with $\mathcal{F} \subseteq \mathcal{G}$ we have $\mathcal{F} = \mathcal{G}$.

Examples 12.21. The principal filter $\mathcal{P}(x)$ is an ultrafilter. To see this, let $\mathcal{F}$ be a filter that contains the filter $\mathcal{P}(x)$. If there were a set S in $\mathcal{F}$ that is not in $\mathcal{P}(x)$: Since S is not in $\mathcal{P}(x)$, the set S does not contain x. But $\{x\}$ lies in $\mathcal{P}(x)$, and therefore also in $\mathcal{F}$. Then $S \cap \{x\} = \emptyset$ were in the filter $\mathcal{F}$ as well–a contradication. We see that there is no such S, and it follows that $\mathcal{F} = \mathcal{P}(x)$. This shows that $\mathcal{P}(x)$ is maximal, hence an ultrafilter.

Examples 12.22. If X is a topological space, the neighborhood filter $\mathcal{O}(x)$ is an ultrafilter if and only if it is equal to $\mathcal{P}(x)$. (This follows from $\mathcal{O}(x) \subseteq \mathcal{P}(x)$.) And, this is the case if and only if $\{x\}$ is open in X.

Proposition 12.23. *A filter $\mathcal{F}$ on a set X is an ultrafilter if and only if it has the following property: For all subsets S of X, precisely one of the sets S and $X \setminus S$ lies in $\mathcal{F}$.*

Proof. One direction is easy: A filter with that property is an ultrafilter. To see that, let $\mathcal{G}$ be a filter with $\mathcal{F} \subseteq \mathcal{G}$. If there were a set $G \subseteq X$ in $\mathcal{G} \setminus \mathcal{F}$, then $X \setminus G \in \mathcal{F}$ by assumption. But then $G \in \mathcal{G}$ and $X \setminus G \in \mathcal{G}$, a contradiction.

Let us now assume that $\mathcal{F}$ is an ultrafilter, and let $S \subseteq X$ be any subset.

If $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ then

$$\mathcal{B} = \{ S \cap F \,|\, F \in \mathcal{F} \}$$

is (!) a basis for a filter $\mathcal{G}$ on X. Because we have $S \cap F \subseteq F$, this filter contains $\mathcal{F}$. And since $S \cap F \subseteq S$, it eventually lies in S. Because $\mathcal{F}$ is an ultrafilter, we deduce $\mathcal{F} = \mathcal{G}$, so that $S \in \mathcal{F}$.

Replacing the set S by its complement $X \setminus S$, we see: If $(X \setminus S) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, then $X \setminus S \in \mathcal{F}$.

It remains to show that one of these two is always the case: If $S \subseteq X$ is a subset such that $S \cap F_1 = \emptyset$ and $(X \setminus S) \cap F_2 = \emptyset$ for some pair of subsets $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \subseteq (X \setminus S) \cap S = \emptyset$, which is not possible. □

The following result implies the existence of enough ultrafilters.

Proposition 12.24. *Every filter is contained in an ultrafilter.*

Proof. Given a totally ordered chain of filters on a set X, their union is also a filter on X. The result now follows from Zorn's Lemma. □

12.7 Some discussion

The presentation so far might suggest that filters are (even) more general than nets. However, that is not true: If $\mathcal{F}$ is any filter on X, then $\mathcal{F}$ is a directed set when we define $F \leqslant F'$ as $F \supseteq F'$. Since the $F \in \mathcal{F}$ are not empty, we can choose an element x_F in F for each F in $\mathcal{F}$ and obtain a net $(x_F \,|\, F \in \mathcal{F})$ in X. If S is in $\mathcal{F}$, then this net is eventually in S (for any choice of points), so that $\mathcal{F} \subseteq \mathcal{F}(x_F)$. Thus, if $\mathcal{F}$ converges to a point x_∞, so does that net.

Exercise 39. Let $X = \{\alpha, \omega\}$ with the topology $\{\emptyset, \{\omega\}, X\}$. Describe the filters $\mathcal{O}(x)$ and $\mathcal{P}(x)$ for all point x in the space X. Are there other filters on X? Which filters on X are ultrafilters? For each filter on X find all of its limits.

13 Topology with filters

We can now use the language introduced in the previous section to rephrase some of the main topological notions in terms of nets and (ultra)filters.

13.1 Closure

We will now characterize the closure operator in terms of limits of filters.

Theorem 13.1. *Let X be a topological space, and let $S \subseteq X$ be a subset. Then x is in the closure $\overline{S}$ of S in X if and only of x is a limit of a filter $\mathcal{F}$ that eventually lies in S.*

Proof. A point x is in the closure of S if and only if every open neighborhood of it intersects S.

If x is a limit of a filter $\mathcal{F}$, then $\mathcal{O}(x) \subseteq \mathcal{F}$. If in addition $S \in \mathcal{F}$, then $S \cap U \in \mathcal{F}$ for all $U \in \mathcal{O}(x)$, so that these sets are not empty, and x in $\overline{S}$.

Conversely, if x lies in $\overline{S}$, then

$$\mathcal{B} = \{U \cap S \,|\, U \in \mathcal{O}(x)\}$$

is (!) a basis for a filter $\mathcal{F}$ on X, and that filter contains $\mathcal{O}(x)$, so that the filter $\mathcal{F}$ converges to x. □

Corollary 13.2. *A subset A of a topological space X is closed if and only it it contains the limits of all filters on X that eventually lie in A.*

Note that we could have replaced 'filter' by 'ultrafilter' in both of these results.

13.2 Continuity

Let $\varphi\colon X \to Y$ be a map. Every sequence (or net) in X gives rise to a sequence (or net) in Y by composition. There is a similar construction for filters.

Definition 13.3. Let $\mathcal{F}$ be a filter on X. Then

$$\{G \subseteq Y \,|\, \varphi^{-1}G \in \mathcal{F}\}$$

is (!) a filter on Y, the *image* $\varphi(\mathcal{F})$ of $\mathcal{F}$ under φ.

Proposition 13.4. *The set*

$$\mathcal{B} = \{\, \varphi(F) \,|\, F \in \mathcal{F} \}$$

is a basis for $\varphi(\mathcal{F})$*, so that*

$$\varphi(\mathcal{F}) = \{\, G \subseteq Y \,|\, G \supseteq \varphi(F) \text{ for some } F \in \mathcal{F} \}.$$

Proof. It is easy to see that $\mathcal{B}$ is a basis, so it generates the filter on the right hand side of the second equation. It remains to explain that equality.

If $G \supseteq \varphi(F)$ for some $F \in \mathcal{F}$, then we get $\varphi^{-1}G \supseteq \varphi^{-1}\varphi(F) \supseteq F$. It follows that $\varphi^{-1}G \in \mathcal{F}$.

Conversely, if $\varphi^{-1}G \in \mathcal{F}$, then $G \supseteq \varphi(\varphi^{-1}G) = \varphi(F)$ for $F = \varphi^{-1}G \in \mathcal{F}$. □

Proposition 13.5. *If* $\varphi\colon X \to Y$ *is a map, and if* $\mathcal{F}$ *is an ultrafilter on* X*, then* $\varphi(\mathcal{F})$ *is an ultrafilter on* Y.

Proof. If $T \subseteq Y$ is a subset, then

$$\varphi^{-1}(Y \setminus T) = X \setminus \varphi^{-1}T.$$

Therefore, if we know that one of $\varphi^{-1}T$ and $X \setminus \varphi^{-1}T$ is in $\mathcal{F}$, then we can deduce that one of T and $Y \setminus T$ is in $\varphi(\mathcal{F})$. □

Now that we know how to map filters, we can finally characterize continuity of maps between topological spaces in terms of limits of filters.

Theorem 13.6. *A map* $\varphi\colon X \to Y$ *between topological spaces is continuous if and only if* $\varphi(x)$ *is a limit of* $\varphi(\mathcal{F})$ *for every filter* $\mathcal{F}$ *on* X *that converges to a point* x *in* X.

Proof. One direction is easy: If $\varphi\colon X \to Y$ is continuous, and x is a limit of $\mathcal{F}$, then this means that $\mathcal{O}(x) \subseteq \mathcal{F}$. We have to show that $\mathcal{O}(\varphi(x)) \subseteq \varphi(\mathcal{F})$. To do so, let us take $V \in \mathcal{O}(\varphi(x))$. Then $\varphi^{-1}V \in \mathcal{O}(x)$ by continuity. It follows that $\varphi^{-1}V \in \mathcal{F}$, hence $V \in \varphi(\mathcal{F})$. Since V was arbitrary, we have proven the desired inclusion.

For the other direction, let us assume that $B \subseteq Y$ is closed. Then we have to show that $\varphi^{-1}B \subseteq X$ is closed. We can use the characterization of closedness in terms of limits. So we assume that we have a filter $\mathcal{F}$ on X that contains $\varphi^{-1}B$ and converges to some point x. We would like to show that $x \in \varphi^{-1}B$. By assumption on φ we know that $\varphi(x)$ is a limit of $\varphi(\mathcal{F})$. And $\varphi^{-1}B \in \mathcal{F}$ means $B \in \varphi(\mathcal{F})$. Since B is closed, this implies $\varphi(x) \in B$, so that $x \in \varphi^{-1}B$, as desired. □

13.3 Hausdorff spaces

We will now characterize Hausdorff spaces in terms of limits of ultrafilters.

Theorem 13.7. *A topological space is a Hausdorff space if and only if every (ultra)filter on it has at most one limit.*

Proof. Let X be a Hausdorff space, and assume that $p, q \in X$ are limits of the same filter $\mathcal{F}$. If p and q were different, we could find disjoint open neighborhoods $U(p) \in \mathcal{O}(p) \subseteq \mathcal{F}$ and $U(q) \in \mathcal{O}(q) \subseteq \mathcal{F}$. But that cannot happen since we are assuming $\mathcal{O}(p) \cup \mathcal{O}(q) \subseteq \mathcal{F}$. So limits of filters are unique.

Conversely, let us assume that X is not a Hausdorff space. Then we can find two points $p \neq q$ in X that cannot be separated by open subsets. This means that none of the sets $U \cap V$ for $U \in \mathcal{O}(p)$ and $V \in \mathcal{O}(q)$ would be empty. Then

$$\mathcal{B} = \{\, U \cap V \,|\, U \in \mathcal{O}(p) \text{ and } V \in \mathcal{O}(q) \,\}$$

is (!) a basis for a filter $\mathcal{F}$ on X. By construction, we have $\mathcal{O}(p) \cup \mathcal{O}(q) \subseteq \mathcal{F}$, so that $\mathcal{F}$ converges to both p and q. □

13.4 Compact spaces

We will now characterize compact spaces in terms of limits of ultrafilters.

Theorem 13.8. *A topological space is compact if and only if every ultrafilter on it has at least one limit.*

Proof. Let us first assume that X is compact, and let $\mathcal{F}$ be an ultrafilter on X. If no point x in X is a limit of $\mathcal{F}$, then $\mathcal{O}(x) \nsubseteq \mathcal{F}$ for all x in X. This means that we can find, for each x in X, an open neighborhood $U(x)$ that does not lie in $\mathcal{F}$. Since $\mathcal{F}$

is an ultrafilter, we must have $X \setminus U(x) \in \mathcal{F}$. But X is compact, so that

$$X = \bigcup_{x \in X_0} U(x)$$

for some finite subset $X_0 \subseteq X$. It follows that

$$\emptyset = \bigcap_{x \in X_0} X \setminus U(x) \in \mathcal{F},$$

a contradiction.

Let us now assume that X is not compact, so that there is an open cover $\mathcal{C}$ that does not have a finite subcover. This means that for all finite $\mathcal{C}_0 \subseteq \mathcal{C}$, the set

$$X \setminus \bigcup_{U \in \mathcal{C}_0} U$$

is not empty. Then the set

$$\mathcal{B} = \{X \setminus \bigcup_{U \in \mathcal{C}_0} U \mid \mathcal{C}_0 \subseteq \mathcal{C} \text{ finite}\}$$

is (!) a basis for a filter $\mathcal{F}$ on X, and that filter $\mathcal{F}$ is contained in an ultrafilter $\mathcal{G}$ on X. If that ultrafilter had a limit x, then x would be contained in one of the open subsets U in the cover $\mathcal{C}$. So $U \in \mathcal{O}(x)$ and $\mathcal{O}(x) \subseteq \mathcal{G}$ by definition of convergence. But $X \setminus U$ is in $\mathcal{B}$ (take $\mathcal{C}_0 = \{U\}$), so that also $X \setminus U$, and we have arrived at a contradiction. □

Corollary 13.9. *A topological space is a compact Hausdorff space if and only if every ultrafilter on it has a unique limit.*

14 Infinite products and Tychonoff's theorem

We have already seen that a product $X \times Y$ of two compact topological spaces X and Y is again compact. By induction, the same holds for finitely many factors. In this section we will generalize this to infinitely many factors.

14.1 Infinite products

Let $(X_j \mid j \in J)$ be a family of topological spaces X_j, indexed by some set J that may or may not be infinite. We would like to define a topology on the product

$$X = \prod_j X_j$$

such that all the projections $\mathrm{pr}_j \colon X \to X_j$ are continuous, and such that a map $f \colon T \to X$ is continuous if (and only if) all the factors $\mathrm{pr}_j f \colon T \to X_j$ are.

Since the pre-images pr_j^{-1} of open subsets $U_j \subseteq X_j$ need to be open in X, we take these as a subbasis for the *product topology*. This means that their finite (!) intersections

$$\bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(U_j)$$

for finite subsets $J_0 \subseteq J$ form a basis for that topology. In other words, a subset of the product will be open if it has the form

$$\prod_j U_j$$

with $U_j \subseteq X_j$ open *and* $U_j = X_j$ *for almost all* j. (Of course, there will be more open subsets, but those for a basis.) It is easy to see that this topology does precisely what we wanted it to do.

If J is finite, then this is the same as the topology on (finite) products as we have defined it before.

14.2 A characterization of convergence

Let $(X_j \mid j \in J)$ be a family of topological spaces, and let X denote its product with the product topology.

Proposition 14.1. *A filter $\mathcal{F}$ on X converges to a point x if and only if all filters* $\mathrm{pr}_j(\mathcal{F})$ *converge to* $\mathrm{pr}_j(x)$.

Proof. One direction is clear: The continuity of the projections implies that the filter $\mathrm{pr}_j(\mathcal{F})$ converges to the filter $\mathrm{pr}_j(x)$ if $\mathcal{F}$ converges to x.

For the other direction, let $\mathcal{F}$ be a filter on X, and $x \in X$ such that all filters $\mathrm{pr}_j(\mathcal{F})$ converge to $\mathrm{pr}_j(x)$. We need to show that $\mathcal{F}$ converges to x. This means $N \in \mathcal{F}$ for every neighborhood N of x in X. By definition of the product topology, there is a finite set $J_0 \subseteq J$ and open subsets $U_j \in X_j$ for the $j \in J_0$ such that

$$x \in \bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(U_j) \subseteq N.$$

But then $\mathrm{pr}_j(x) \in U_j$ for all $j \in J_0$. By assumption on the convergence in each factor, this implies $U_j \in \mathrm{pr}_j(\mathcal{F})$ for all $j \in J_0$. Equivalently, we have $\mathrm{pr}_j^{-1}(U_j) \in \mathcal{F}$ for all $j \in J_0$. Because J_0 is finite, their intersection is in $\mathcal{F}$, and then N is in $\mathcal{F}$ as well. □

14.3 Tychonoff's theorem

We have seen before that a product $X \times Y$ of two compact topological spaces X and Y is again compact. By induction, the same holds for finitely many factors. We can now generalize this to arbitrary products.

Theorem 14.2. (Tychonoff) *Let $(X_j \mid j \in J)$ be a family of compact topological spaces. Then their product X is compact with respect to the product topology.*

Proof. Let $\mathcal{F}$ be an ultrafilter on X. We would like to show that it converges. The images $\mathrm{pr}_j(\mathcal{F})$ are filters on the X_j. They are actually ultrafilter as well. (Exercise!) By hypothesis, they each converge to some $x_j \in X_j$. The previous proposition now implies that $\mathcal{F}$ converges to $x = (x_j)$. □

Chapter V

Mapping spaces

15 The compact-open topology

If S and T are sets, so is the set $\mathrm{Map}(S,T)$ of maps $S \to T$. If V and W are two vector spaces, so is the set $\mathrm{Hom}(V,W) \subseteq \mathrm{Map}(S,T)$ of linear maps $V \to W$. So, if X and Y are topological spaces, so should be the set $\mathcal{C}(X,Y) \subseteq \mathrm{Map}(X,Y)$ of continuous maps. We will now see how that can be achieved.

15.1 The compact-open topology

Let X and Y be topological spaces. If $K \subseteq X$ is compact, and $V \subseteq Y$ is open, we have the subset

$$\mathrm{S}(K,V) = \{ f \in \mathcal{C}(X,Y) \mid f(K) \subseteq V \}$$

of $\mathcal{C}(X,Y)$. These form (!) a subbasis for a topology on $\mathcal{C}(X,Y)$.

Definition 15.1. The topology generated by

$$\{ \mathrm{S}(K,V) \mid K \subseteq X \text{ compact, } V \subseteq Y \text{ open} \}$$

is the *compact-open topology* on $\mathcal{C}(X,Y)$.

A subset $U \subseteq \mathcal{C}(X,Y)$ is open if and only for each $f \in U$ there are finitely many compact $K_j \subseteq X$ and open $V_j \subseteq Y$ such that

$$f \in \bigcap_{j \in J} \mathrm{S}(K_j, V_j) \subseteq U.$$

Examples 15.2. Let X be discrete. Since every map $X \to Y$ is continuous, we can identify $\mathcal{C}(X,Y)$ with

$$\prod_{x \in X} Y.$$

The compact subspaces $K \subseteq X$ are precisely the finite ones, and

$$\mathrm{S}(K,V) = \bigcap_{x \in K} \mathrm{S}(\{x\},V) \cong \bigcap_{x \in K} \mathrm{pr}_x^{-1} V.$$

It follows that the compact-open topology is the product topology, even if X is infinite.

15.2 First properties

Proposition 15.3. *Let $e\colon X' \to X$ and $g\colon Y \to Y'$ be two continuous maps. Then the maps*

$$g_*\colon \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Y'),\, f \longmapsto gf$$
$$e^*\colon \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X',Y),\, f \longmapsto fe$$

are continuous.

Proof. This follows from

$$(g_*)^{-1}\mathrm{S}(K,V') = \mathrm{S}(K,g^{-1}V')$$
$$(e^*)^{-1}\mathrm{S}(K',V) = \mathrm{S}(e(K'),V).$$

Note that we have used that $g^{-1}V$ is open and that $e(K')$ is compact. □

Exercise 40. Use this and the 'universal' constant map $X \to \star$ to show that the map

$$Y \longrightarrow \mathcal{C}(X,Y)$$

that sends y to the constant map with value y is continuous for all spaces X and Y.

Proposition 15.4. *If Y is a Hausdorff space, so is $\mathcal{C}(X,Y)$ for all topological spaces X.*

Proof. If $f \neq g$, then $f(x) \neq g(x)$ for some x in X. We can separate these point of Y with open subsets U and V. Then $\mathrm{S}(\{x\},U)$ and $\mathrm{S}(\{x\},V)$ separate f from g. □

Exercise 41. Show the converse: If $\mathcal{C}(X,Y)$ is a Hausdorff space for all topological spaces X, then so is Y.

15.3 An example

Let $I = [0,1]$ be the unit interval. This a compact Hausdorff space. We already know that the space $\mathcal{C}(I,I)$ of self-maps $I \to I$ is also a Hausdorff space; it is even metric. But we will now see, following an argument of Groves [Gro12], that it is not compact.

For each $t \in I$ let $U(t) \subseteq \mathcal{C}(I,I)$ be the open set

$$U(t) = \mathrm{S}(\{x\}, \mathrm{B}(x, 1/23))$$

that consists of the continuous functions f such that

$$x - 1/23 < f(x) < x + 1/23.$$

Since every continuous self-map of I has a fixed point, these sets cover the whole space $\mathcal{C}(I,I)$. On the other hand, there is no finite subset $I_0 \subseteq I$ such that $\{\, U(t) \,|\, t \in I_0 \,\}$ is a finite subcover: Draw a picture!

Exercise 42. This one is not straightforward, but nevertheless worth trying: Show that there does not exist a continuous (!) map

$$\Phi\colon \mathcal{C}(I,I) \longrightarrow I$$

that picks a fixed point for each continuous self-map: No continuous Φ satisfies

$$f(\Phi(f)) = \Phi(f)$$

for all f. For a solution, see [Szy15].

16 Local compactness

Let X be a topological space. Consider the following properties.

(1) For every point x in X there is a compact neighborhood $K \subseteq X$ of the point x.
(2) For every point x in X and every open neighborhood U of x there is a compact neighborhood $K \subseteq X$ of x such that $K \subseteq U$.
(3) For every point x in X and every neighborhood N of x there is a compact neighborhood $K \subseteq X$ of x such that $K \subseteq N$.

Exercise 43. Convince yourself that the properties and (2) and (3) are equivalent, and that they imply property (1).

Definition 16.1. A topological space that satisfies property (3)–and therefore (2) and (1)–is called *locally compact*.

Examples 16.2. Every discrete space is locally compact, and the space $\mathbb{R}^n$ is locally compact for all $n \geqslant 0$.

16.1 Compact Hausdorff spaces again

For Hausdorff spaces, the definition of local compactness becomes simpler:

Proposition 16.3. *If X is a Hausdorff space, the property* (1) *is equivalent to* (2) *and* (3).

Proof. Let U be an open neighborhood of x in X. By assumption, the point x lies in some compact subspace L of X. Since X is Hausdorff, the subset L is closed in X. Since U is open in X, the subset $U \cap L$ is open in L, so that its complement

$$A = L \setminus (U \cap L)$$

is closed in L, and therefore compact. The set A does not contain x, because x lies in $U \cap L$. By compactness, we can therefore find disjoint open subsets V and W such that $x \in V$ and $A \subseteq W$. Then

$$K = L \setminus W$$

will be our K:

First of all, the set K is closed in L, hence compact. Then x lies in K, since it lies in L, but not in W. And we have

$$K = L \setminus W \subseteq L \setminus A = U \cap L \subseteq U.$$

Finally, we get $L \cap V \subseteq L \setminus W = K$, and this shows that K is a neighborhood, for so are L and V and therefore also their intersection. □

Corollary 16.4. *A compact Hausdorff space is locally compact.*

16.2 An application to compact-open topologies

We can now show that the compact open topology is metrizable if the target is metrizable and the source is a compact Hausdorff space.

Theorem 16.5. *Let X be a compact Hausdorff space, and let Y be a metric space. Then the compact-open topology on $\mathcal{C}(X,Y)$ can be defined by the metric*

$$d(f,g) = \sup\{\, d(f(x),g(x)) \mid x \in X \,\}$$

of uniform convergence.

Proof. Let f lie in a subset of the form $\mathrm{S}(K,V)$. We would like to find a ball around f that is contained in $\mathrm{S}(K,V)$. We know that $f(K) \subseteq V$. Since V is open, for all y in $f(K)$ there is an $\varepsilon(y)$ such that $\mathrm{B}(y,\varepsilon(y)) \subseteq V$. The sets $\mathrm{B}(y,\varepsilon(y)/2)$, for y in Y, form an open cover of the compact space $f(K)$. Then there is a finite subset $Y_0 \subseteq Y$ such that

$$f(K) \subseteq \bigcup_{y \in Y_0} \mathrm{B}(y,\varepsilon(y)/2).$$

Set

$$\varepsilon = \min\{\, \varepsilon(y) \mid y \in Y_0 \,\}.$$

Now, if g is a function $X \to Y$ such that $d(f,g) < \varepsilon/2$, then we can show $g(K) \subseteq V$ as well, so that $\mathrm{B}(f,\varepsilon/2) \subseteq \mathrm{S}(K,V)$: If x is in K, then $d(f(x),g(x)) < \varepsilon/2$, and

$$d(f(x),y) < \varepsilon(y)/2 < \varepsilon/2$$

for some $y \in Y_0$. It follows from the triangle inequality that

$$d(g(x),y) < \varepsilon,$$

and this implies $g(x) \in V$.

Conversely, given f and $\varepsilon > 0$, we need to show that around each point g of the ball $\mathrm{B}(f,\varepsilon)$ there are finitely many sets of the form $\mathrm{S}(K,V)$ such that their intersection is contained in $\mathrm{B}(f,\varepsilon)$ as well. It actually suffices to do that for $g = f$ itself, because the general case follows by re-centering, as in the first chapter.

Given x in X, we know that $f^{-1}\mathrm{B}(f(x),\varepsilon/2)$ is an open neighborhood of x. Since compact Hausdorff spaces are locally compact, this open neighborhood contains a compact neighborhood $K(x)$, so that

$$f(K(x)) \subseteq \mathrm{B}(f(x),\varepsilon/2)$$

for all x in X. Since X is compact, finitely many of these compact neighborhoods cover X:

$$X = \bigcup_{x\in X_0} K(x)$$

for a finite subset $X_0 \subseteq X$. Now we already have

$$f \in \bigcap_{x\in X_0} \mathrm{S}(K(x),\mathrm{B}(f(x),\varepsilon/2)).$$

It therefore suffices to see that this intersection is contained in $\mathrm{B}(f,\varepsilon)$. To do so, let $g\colon X \to Y$ be in that intersection as well. If we then have x in X, we have $x \in K(x_0)$ for an x_0 in X_0, and this implies

$$f(x) \in \mathrm{B}(f(x_0),\varepsilon/2)$$

by the above, and

$$g(x) \in \mathrm{B}(f(x_0),\varepsilon/2)$$

by assumption. The triangle inequality implies $d(f(x),g(x)) < \varepsilon$. Since this holds for all x, we get $d(f,g) < \varepsilon$, as claimed. □

There is a more general statement for locally compact sources, but we won't go further into this.

17 Adjoint constructions

Given sets X, Y, and Z, there is a natural bijection

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Namely, given a map

$$f\colon X \times Y \longrightarrow Z,$$

the map

$$f^\sharp\colon X \longrightarrow \mathrm{Map}(Y, Z)$$

sends x to the map

$$f^\sharp(x)\colon Y \to Z$$

that sends y to $f(x,y)$. We will often write $f^\sharp_x$ instead of $f^\sharp(x)$, because

$$f^\sharp_x(y) = f(x,y)$$

is easier to read.

Definition 17.1. The maps f and $f^\sharp$ that correspond under this bijection are called *adjoints* of each other.

We will now explore what happens if X, Y, and Z have topologies, and we pass from the sets $\mathrm{Map}(?,??)$ to the subsets $\mathcal{C}(?,??)$ of continuous maps.

Let $f\colon X \times Y \to Z$ be continuous. Then, for every x in X, the map

$$f^\sharp_x\colon Y \longrightarrow Z, y \longmapsto f(x,y)$$

is (!) continuous. This defines a map

$$f^\sharp\colon X \longrightarrow \mathcal{C}(Y,Z).$$

Theorem 17.2. *For all continuous maps f the adjoint $f^\sharp$ is continuous.*

Proof. It suffices to show that the pre-images of the sets $\mathrm{S}(L,W)$ are open, whenever $L \subseteq Y$ is compact and $W \subseteq Z$ is open. Let x be a point in that pre-image. Then $f(\{x\} \times L) \subseteq W$. Because f is continuous and L is compact, there is even an open neighborhood U of x in X with $f(U \times L) \subseteq W$. This implies that the entire set U lies in the pre-image, so that this is open. □

Adjunction defines a map

$$\mathcal{C}(X \times Y, Z) \longrightarrow \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

It is always injective, and we may wonder under which assumptions (if any) it is surjective, hence bijective.

Theorem 17.3. *If Y is locally compact, the adjunction map*

$$\mathcal{C}(X \times Y, Z) \longrightarrow \mathcal{C}(X, \mathcal{C}(Y, Z))$$

is bijective.

Proof. It remains to prove surjectivity. For that purpose, let $f\colon X \times Y \to Z$ be a map such that its adjoint $f^\sharp\colon X \to \mathcal{C}(Y,Z)$ is continuous. We will show that f is continuous around each point (x,y).

Since $f^\sharp_x$ is continuous, and Y is locally compact, for each open neighborhood W of $f(x,y)$ in Z there is a compact neighborhood L of y in Y such that $f^\sharp_x(L) \subseteq W$. Since $f^\sharp$ is continuous, the subset

$$U = (f^\sharp)^{-1}\mathrm{S}(L,W) = \{x' \in X \mid f^\sharp_{x'}(L) \subseteq W\}$$

is an open neighborhood of x in X. And then $U \times L$ is a neighborhood of (x,y) such that $f(U \times L) = f^\sharp(U)(L) \subseteq W$. □

Source and target of the adjunction map carry compact-open topologies, and we wonder further if the adjunction map is continuous or even a homeomorphism. We will not go into this here.

We end with some simple consequences of the preceding results that illustrate how to work with adjoints.

Corollary 17.4. *If X is locally compact, them the evaluation map*

$$\mathrm{ev}_X\colon \mathcal{C}(X,Y) \times X \longrightarrow Y,\ (f,x) \longmapsto f(x)$$

is continuous.

Proof. Its adjoint is the identity map on $\mathcal{C}(X,Y)$. □

Corollary 17.5. *If X and Y are locally compact, then the composition map*

$$\mathcal{C}(Y,Z)\times\mathcal{C}(X,Y)\longrightarrow\mathcal{C}(X,Z),\ (g,f)\longmapsto gf$$

is continuous.

Proof. It is enough to show continiouity of the adjoint. The diagram

$$\begin{array}{ccc} \mathcal{C}(Y,Z)\times\mathcal{C}(X,Y)\times X & \xrightarrow{\qquad\qquad\qquad} & Z \\ {\scriptstyle \mathrm{id}\times\mathrm{ev}_X}\searrow & & \nearrow{\scriptstyle \mathrm{ev}_Y} \\ & \mathcal{C}(Y,Z)\times Y & \end{array}$$

shows that the adjoint is a composition of two continuous maps. □

Corollary 17.6. *Let $p\colon X\to X'$ be an identification. Then for all locally compact spaces Y the product*

$$p\times\mathrm{id}\colon X\times Y\longrightarrow X'\times Y$$

is an identification as well.

Proof. Let T be a topological space, and let $f\colon X'\times Y\to T$ be a map such that the composition $f(p\times\mathrm{id})$ is continuous.

$$\begin{array}{ccc} X\times Y & \xrightarrow{p\times\mathrm{id}} & X'\times Y \\ & {\scriptstyle f(p\times\mathrm{id})}\searrow & \downarrow{\scriptstyle f} \\ & & T, \end{array}$$

We have to show that f is continuous.

The composition $f^\sharp p$ in the diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ & {\scriptstyle f^\sharp p}\searrow & \downarrow{\scriptstyle f^\sharp} \\ & & \mathcal{C}(Y,T), \end{array}$$

is continuous because it is adjoint to $f(p\times\mathrm{id})$. Since p is an identification, this implies that $f^\sharp$ is continuous. Then the continiouity of f follows by adjunction again. □

Exercise 44. Give conditions under which the product $p\times q\colon X\times Y\to X'\times Y'$ of two identifications $p\colon X\to X'$ and $q\colon Y\to Y'$ is again an identification.

Bibliography

[B40] N. Bourbaki. Topologie générale. Hermann & Cie., 1940 and later.

[B93] G.E. Bredon. Topology and geometry. Springer-Verlag, New York, 1993.

[tD08] T. tom Dieck. Algebraic topology. European Mathematical Society, Zürich, 2008.

[Gro12] J. Groves. An elementary counterexample in the compact-open topology. Am. Math. Mon. 119 (2012) 693–694.

[LS15] G. Laures, M. Szymik. Grundkurs Topologie. 2. Aufl. Springer Spektrum, Heidelberg, 2015.

[M00] J.R. Munkres. Topology. 2nd ed. Prentice Hall, 2000.

[SS78] L.A. Steen, J.A. Seebach. Counterexamples in Topology. Reprint of the second (1978) edition. Dover Publications, 1995.

[Szy15] M. Szymik. Homotopies and the universal fixed point property. Order 32 (2015) 301–311.