Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4190 Manifolds
Spring 2015

Solutions
Exam May 18, 2015

1 **a)** A smooth n-dimensional manifold is a pair $(M, \mathcal{A})$ where M is a second countable Hausdorff space, and $\mathcal{A}$ is a maximal smooth atlas on M. The latter means the following. The collection $\mathcal{A}$ is an *atlas* on M if, for every point $p \in M$, there is a chart $(x, U) \in \mathcal{A}$ consisting of an open subset U of M containing p and a homeomorphism $x \colon U \to U'$ where U' is an open subset of $\mathbb{R}^n$. This atlas is *smooth* if for any $(x, U) \in \mathcal{A}$ and $(y, V) \in \mathcal{A}$ the map $y \circ x^{-1} \colon x(U \cap V) \to y(V)$ is smooth. The atlas $\mathcal{A}$ is *maximal* if for $(x, U) \in \mathcal{A}$ and any chart (y, V) such that $y \circ x^{-1}$ is smooth, then $(y, V) \in \mathcal{A}$.
(2 Points)

b) A map $f \colon (M, \mathcal{A}) \to (N, \mathcal{B})$ is smooth if f is continuous and for any $(x, U) \in \mathcal{A}$ and $(y, V) \in \mathcal{B}$ the map $y \circ f \circ x^{-1} \colon x(U \cap f^{-1}(V)) \to y(V)$ is smooth.
(2 Points)

c) Let $\mathcal{A}$ be an atlas on M. Then for each $(x, U) \in \mathcal{A}$ we define a chart

$$\tilde{x} \colon \pi^{-1}(U) \xrightarrow{h_x} U \times \mathbb{R}^n \xrightarrow{x \times \mathrm{id}} \mathbb{R}^n \times \mathbb{R}^n, \text{ by } [\gamma] \mapsto (x\gamma(0), (x\gamma)'(0))$$

where π denotes the projection $TM \to M$.
(3 Points)

d) We know that π is continuous. Hence, given a chart (x, U) on M, we need to show that the bottom horizontal map in the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi} & U \\ {\scriptstyle \tilde{x}}\downarrow & & \downarrow{\scriptstyle x} \\ x(U) \times \mathbb{R}^n & \longrightarrow & x(U) \end{array} \tag{1}$$

is smooth. For $(q, v) \in x(U) \times \mathbb{R}^n$, let $[\gamma]$ be the unique element in $\pi^{-1}(U) \subseteq TM$ such that $\tilde{x}([\gamma]) = (q, v)$, i.e., $x\gamma(0) = q$ and $(x\gamma)'(0) = v$. Then we have

$$x \circ \pi \circ \tilde{x}^{-1}(q, v) = x \circ \pi([\gamma]) = x\gamma(0) = q.$$

In other words, the bottom horizontal map in (1) is just the projection onto the first factor. Since the projection is smooth, this shows that π is smooth.
(3 Points)

2 **a)** If f s a diffeomorphism, then $df_p \colon T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$. Hence $\dim M = \dim T_pM = \dim T_{f(p)}N = \dim N$.
(2 Points)

b) Assume that f is not an immersion, i.e., the rank r of f is strictly smaller than the dimension m of M. By the rank theorem, for each $p \in M$, we can choose charts (x, U) of M and (y, V) of N such that $p \in U$, $f(p) \in V$, and

$$y \circ f \circ x^{-1}(q) = (q_1, \ldots, q_r, 0, \ldots, 0)$$

for all $q \in x(U) \in \mathbb{R}^m$. But, since $r < m$, for any sufficiently small $\epsilon > 0$, this implies

$$y \circ f \circ x^{-1}(q_1, \ldots, q_{r+1}, \ldots) = (0, \ldots, 0) = y \circ f \circ x^{-1}(q_1, \ldots, q_{r+1} + \epsilon, \ldots).$$

Hence f is not injective, which contradicts the assumption. Thus f must be an immersion.
(5 Points)

c) We need to show that $M \to f(M)$ is a homeomorphism. It is clearly surjective, and we know by assumption that it is injective. Hence it is a continuous bijective map. But since M is compact and $f(M)$ is Hausdorff ($f(M)$ is a subset of a Hausdorff space), $M \to f(M)$ is also a homeomorphism.
(3 Points)

3

a) We define an appropriate chart $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by

$$\varphi(x, y) = (x, y - f(x)).$$

Since f is smooth, we know that φ is a homeomorphism with smooth inverse $\varphi^{-1}(u, v) = (u, v + f(u))$. Then we have

$$\varphi((\mathbb{R}^m \times \mathbb{R}^n) \cap \Gamma_f) = \varphi(\Gamma_f) = \mathbb{R}^m \times \{0\}.$$

Hence Γ_f is a submanifold of dimension m.
(2 Points)

b) The derivative of f at (x, y, z) is $Df(x, y, z) = (3x^2, 3y^2, 3z^2)$. Hence the only critical point of f is $(0, 0, 0)$. But this point does not lie in the domain of definition. Hence the rank of f is equal to the dimension of the target at every point, which means that f is a submersion.
(2 Points)

c) Define $F \colon \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x, y, z) = (x^3 + y^3 + z^3, x + y + z)$. Then $N = F^{-1}((1, 0))$. Hence, in order to show that N is a submanifold, it suffices to show that $(1, 0)$ is regular value of F.The Jacobian of F is given by

$$J(F) = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{pmatrix}$$

and the critical points of F are those where $J(F)$ has rank < 2. This happens when $x^2 = y^2 = z^2$. But since the points (x, y, z) on N satisfy $x + y + z = 0$, we would have $(x, y, z) = (0, 0, 0)$. But $(0, 0, 0)$ does not satisfy $x^3 + y^3 + z^3 = 1$. Hence the critical points of F do not lie on N, and N is a submanifold of $\mathbb{R}^3$ (of dimension 1).
(3 Points)

d) We consider the determinant as the map $\det\colon M_2(\mathbb{R}) \to \mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Using the chart $M_2(\mathbb{R}) \cong \mathbb{R}^4$ sending A to $\begin{pmatrix} a_{11} & a_{12} & a_{21} & a_{22} \end{pmatrix}$, we have

$$D(\det)(A) = \begin{pmatrix} a_{22} & -a_{21} & -a_{12} & a_{11} \end{pmatrix}.$$

Hence det has rank 1 at all matrices, except the zero-matrix where it has rank and value 0. In particular, 1 is a regular value of det, and $\mathrm{SL}_2(\mathbb{R}) = \det^{-1}(1)$ is a 3-dimensional smooth manifold.
(3 Points)

4 **a)** The derivative of f is given by $Df((x,y)) = \begin{pmatrix} 3x^2 - 6y & 2y - 6x \end{pmatrix}$. For a critical point, $Df(x,y)$ is of rank 0, i.e., both entries vanish. This is the case for $(x,y) = (0,0)$; and for $y = 3x$ and $3x^2 = 6y$, i.e., for $x = 6$ and $y = 18$. The critical values of f are thus $f(0,0) = 0$ and $f(6,18) = -108$.
(3 Points)

b) The derivative of g_a is

$$Dg_a(t) = \begin{pmatrix} 2t \\ 3t^2 - a \end{pmatrix}$$

and the corresponding linear map can only fail to be injective if this matrix is the zero-matrix. This can only be the case if $t = 0$ and $3t^2 = a$. Hence for $a = 0$, $Dg_0(t)$ is not injective for $t = 0$, but for $a \neq 0$ $Dg_a(t)$ is always injective. Hence g_a is an immersion for $a \neq 0$, but it is not an immersion for $a = 0$.
For g_a to be an imbedding it has to be injective. For $a > 0$, we have

$$g_a(\sqrt{a}) = g_a(-\sqrt{a}).$$

This shows that g_a is not injective and hence not an imbedding for $a > 0$.
For $a < 0$, we will show that g_a is an imbedding. Therefor we first show that the composite

$$\tilde{g}_a\colon \mathbb{R} \xrightarrow{g_a} \mathbb{R}^2 \xrightarrow{\mathrm{pr}_2} \mathbb{R},\ t \mapsto t(t^2 - a)$$

is a diffeomorphism onto its image. Because of $a < 0$ the derivative of $\tilde{g}_a$ is strictly positive for all t (it is $\tilde{g}_a'(t) = 3t^2 - a$). The inverse function theorem then implies that $\tilde{g}_a$ is a diffeomorphism onto its image. This shows that g_a has a continuous inverse *defined on the image of* g_a. (To recover t from $g_a(t)$ it suffices to look at the inverse of $\tilde{g}_a$.) Hence g_a is a homeomorphism onto its image. Thus for $a < 0$, g_a is an imbedding.
(1+2+4=7 Points)

5 **a)** A global flow on M is a smooth map

$$\Phi\colon \mathbb{R} \times M \to M$$

such that for all $p \in M$ and $s, t \in R$

- $\Phi(0, p) = p$
- $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$.

A smooth vector field on M is a smooth section of the projection $\pi\colon TM \to M$ of the tangent bundle of M, i.e., a smooth map $\sigma\colon M \to TM$ such that $\pi \circ \sigma = \mathrm{id}_M$.
(2 Points)

b) i) For $(r, \theta) = (1, 0)$ the flow lines are constant.
ii) For $(r, \theta) = (1, \pi/2)$ the flow lines outside the origin are circles; at $z = 0$ the flow lines are constant.
iii) For $(r, \theta) = (1/2, 0)$ the flow lines outside the origin are rays flowing towards the origin; at $z = 0$ the flow lines are constant.
(2 Points)

c) The velocity field of Ψ is

$$S^{2n-1} \to TS^{2n-1},\ z \mapsto [t \mapsto z \cdot e^{it} =: \gamma_z(t)]$$

We need to show that the derivative of the curve $\gamma_z(t)$ is non-zero at $t = 0$. But we have $\gamma_z'(t) = iz \cdot e^{it}$ which is non-zero for all $z \in S^{2n-1}$ and all t.
(3 Points)

d) There are multiple ways to prove this. One is to show that the map

$$S^1 \times \mathbb{R} \to TS^1,\ (z, t) \mapsto [s \mapsto z \cdot e^{it}]$$

is a bundle isomorphism. To show this it suffices to show that it is a linear isomorphism for each fiber. For $z \in S^1$, the map $\mathbb{R} \to T_z S^1$ defined by sending the real number t to the tangent vector at z determined by the curve $[s \mapsto s \cdot tiz]$ which is compatible with the projection maps.
(3 Points)