NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4190 Manifolds**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator (Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College, Hewlett Packard HP30S) is allowed.

Language: English

Number of pages: 2

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Give the definition of a smooth n-dimensional manifold.

b) Let M and N be smooth manifolds. Give the definition of a smooth map $f\colon M \to N$.

c) The tangent space TM of a smooth n-dimensional manifold M is a smooth $2n$-dimensional manifold. Using the charts of M, describe the charts of TM.

d) Show that the map $\pi\colon TM \to M$ given by $\pi([\gamma]) = \gamma(0)$ is smooth.

Problem 2

a) Show that if two smooth manifolds are diffeomorphic, then they have the same dimension.

b) Let M and N be smooth manifolds, and let $f\colon M \to N$ be a smooth map of constant rank. Show that if f is injective, then f is an immersion.

c) Let M be a compact smooth manifold, and let $g\colon M \to N$ be an injective immersion. Show that g is an imbedding.

Problem 3

a) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map, and let

$$\Gamma_f = \{(x, f(x)) | x \in \mathbb{R}^m\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

be the graph of f. Show that Γ_f is a submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. What is the dimension of Γ_f?

b) Define the map $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$f(x, y, z) = x^3 + y^3 + z^3.$$

Show that f is a submersion.

c) Let N be the subset of points (x, y, z) in $\mathbb{R}^3$ which satisfy the two equations

$$x^3 + y^3 + z^3 = 1, \quad (1)$$

$$x + y + z = 0. \quad (2)$$

Show that N is a submanifold of $\mathbb{R}^3$.

d) Show that the special linear group $\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) | \det(A) = 1\}$ is a 3-dimensional smooth manifold.

Problem 4

a) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find the critical values of f.

b) For $a \in \mathbb{R}$, define $g_a\colon \mathbb{R} \to \mathbb{R}^2$ by

$$g(t) = (t^2, t(t^2 - a)).$$

Discuss whether g_a is an immersion and whether g_a is an imbedding for
i) $a = 0$,
ii) $a > 0$,
iii) $a < 0$.

Problem 5

a) Let M be a smooth manifold. Give a definition of a global flow and of a smooth vector field on M.

b) For $(r, \theta) \in \mathbb{R}^2$, define the flow $\Phi\colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ by

$$\Phi(t, z) = z \cdot r^t e^{it\theta}.$$

Describe the flow lines for

i) $(r, \theta) = (1, 0)$, ii) $(r, \theta) = (1, \pi/2)$, iii) $(r, \theta) = (1/2, 0)$.

c) For any $n \geq 1$, define a flow on the odd-dimensional sphere $S^{2n-1} \subset \mathbb{C}^n$ by

$$\Psi(t, z) = z \cdot e^{it}.$$

Show that the velocity field of Ψ does not vanish on S^{2n-1}.

d) Show that the tangent bundle TS^1 on the circle S^1 is trivial.