TMA4190 Manifolds
Some exercises

Spring 2015

This is a list of some exercises you may want to think about while you prepare for the exam on May 18. We will discuss some of the problems in the exercise class on Tuesday, May 5, 16:15-18:00 in seminar room 734 in Sentralbygg 2. For any further questions, please send me an email to gereon.quick@math.ntnu.no and/or come to my office hours on Mondays and Wednesdays 15:00-16:00 in 1246 in Sentralbygg 2.

Problem 1. Let $f\colon M \to N$ be a smooth map between smooth manifolds. Make sure you know what it means that f is

(1) an embedding
(2) injective
(3) an immersion
(4) a submersion
(5) a diffeomorphism.

Which of the previous properties implies which other properties? Can you think of examples of maps which show that the reverse implications are not true?

Problem 2. Let x, y, z, w be the standard coordinates of $\mathbb{R}^4$. Is the set of points in $\mathbb{R}^4$ which satisfy the equation

$$x^5 + y^5 + z^5 + w^5 = 1$$

a manifold? Explain your answer! (Assume that the subset is given the subspace topology in $\mathbb{R}^4$. (Make sure all these terms have a meaning for you!))

Problem 3. Let N be the subset of points (x, y, z) in $\mathbb{R}^3$ which satisfy the two equations

$$x^3 + y^3 + z^3 = 1, \tag{1}$$

$$z = xy. \tag{2}$$

Show that N is a submanifold of $\mathbb{R}^3$.

Problem 4. Show that the composition of two maps of constant rank need not have constant rank.

Problem 5. Let M and N be smooth manifolds, and let $f\colon M \to N$ be a (smooth) submersion. Show that f is an open map.

Problem 6. Let M be a non-empty smooth compact manifold. Show that there is no smooth submersion $f\colon M \to \mathbb{R}^k$ for any $k > 0$.

Problem 7. Consider the map $f\colon (-\pi/2, 3\pi/2) \to \mathbb{R}^2$ defined by

$$f(t) = (\cos t, \sin 2t).$$

Show that f is an injective smooth immersion. Is f also an embedding?

Problem 8. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^3 + xy + y^3.$$

Discuss for which values $a \in \mathbb{R}$ the set $f^{-1}(a)$ is a submanifold of $\mathbb{R}^2$.

Problem 9. Show that the orthogonal group $\mathrm{O}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$.

Problem 10. Let $f\colon M \to N$ be a smooth map with constant rank and let y be a regular value of f. Show that there is an isomorphism

$$T_p(f^{-1}(y)) \cong \operatorname{Ker} T_p f.$$

for all $p \in f^{-1}(y)$.