Exam problems for TMA4190 Manifolds
Thursday 9 June, 2016

SOLUTIONS

Problem 1 We consider smooth functions $F : M \to N$ between smooth manifolds M og N without boundary. .

(a) What is the condition for F to be an immersion, and what does it mean that F is a submersion ? Give an example in both cases.

(b) Give an example where F is an injective immersion, but is not an embedding of M into N. If M is compact and F is an injective immersion, must $F(M)$ be a submanifold of N? Justify your answer.

(c) If F is a diffeomorphism, explain (possibly by categorical reasoning, without coordinates) why the induced map of tangent bundles, $dF : TM \to TN$, viewed as a map between manifolds, is also a diffeomorphism.

Solutions

(a) F is an immersion if the rank of F equals $\dim M$ at each point $p \in M$, or equivalently, the linear map $dF|_p$ from T_pM to $T_{F(p)}N$ is injective for each p. Similarly, F is a submersion if the rank of F equals $\dim N$ at each point $p \in M$, or equivalently, $dF|_p$ is surjective. A simple example of an immersion is the inclusion of $\mathbb{R}$ into $\mathbb{R}^2$, $x \to (x, 0)$, and for a submersion we can take the projection $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \to x$.

(b) To construct an example of an injective immersion which is not an embedding, we mean an example where the image $F(M)$ is not a submanifold of N. This happens when $F(M)$ with the subspace topology from N is not homeomorphic to M. For example, one can consider a smoothly parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that the velocity vector $d\gamma/dt$ never vanishes (to make γ an immersion, and with γ injective (so that it becomes an injective immersion), and finally so that, for example, $\lim \gamma(t) = \gamma(0)$ as $t \to \infty$. Then the image curve $\gamma(\mathbb{R})$ as subspace of $\mathbb{R}^2$ is not even locally Euclidean around the point $\gamma(0)$.

But if M is compact, then (by point set topology) the continuous and bijective map $F : M \to F(M)$ has a continuous inverse and hence is a homeomorphism. Therefore, for an injective immersion $F : M \to N$, the image $F(M)$ will be a (regular) submanifold of N.

(c) Note that the tangent map construction $dF : TM \to TN$ has the property that for a composition $G \circ F : M \to N \to P$, we have $d(G \circ F) = dG \circ DF$, and also dF is the identity Id on TM if F is the identity Id on M. Hence, if F is a diffeomorphism, so that F^{-1} exists, then dF has inverse map $(dF)^{-1} = d(F^{-1})$ since $d(F^{-1}) \circ dF = d(F^{-1} \circ F) = d(Id) = Id$.

Problem 2 We consider the following subsets (quadratic hypersurfaces) of the Cartesian 4-space $\mathbb{R}^4$ with coordinates $x_i, 1 \leq i \leq 4$, determined by the quadratic equation

$$M_k : x_1x_4 - x_2x_3 = k \ , \ k \in \mathbb{R}$$

(a) Show that for $k \neq 0$, M_k is a closed 3-dimensional submanifold of $\mathbb{R}^4$. Is M_k compact? Justify your answer.

(**b**) Show that all M_k, $k \neq 0$, are diffeomorphic to each other, and their union $\tilde{M}$ is an open subset of $\mathbb{R}^4$.

(c) We introduce three vector fields X_i on $\mathbb{R}^4$ by setting

$$\begin{aligned} X_1 &= x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4}, \ X_2 = x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2}, \\ X_3 &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} \end{aligned}$$

Show that X_1, X_2, X_3 are linearly independent at each point on $\tilde{M}$.

(d) Given a vector field X on $\mathbb{R}^4$ (with suitable restrictions), by integration of the dynamical system

$$\frac{d\gamma}{dt} = X|_{\gamma(t)}$$

one obtains the integral curves $\gamma(t) = (x_1(t), .., x_4(t))$, with $\gamma(0) = \mathbf{x}$, expressed as a global flow

$$\theta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4, \quad \gamma(t) = \theta(t, \mathbf{x}) = \theta_t(\mathbf{x}),$$

where each $\theta_t : \mathbb{R}^4 \to \mathbb{R}^4$ is an invertible transformation. For "linear" vector fields, such as X_i in item (c), θ_t is also a linear map. The family $\{\theta_t, t \in \mathbb{R}\}$ is called a 1-parameter group of transformations since $\theta_0 = Id$ and $\theta_s\theta_t = \theta_{s+t}$.

Determine the 1-parameter groups $\{\theta_t^{(i)}\}$, $i = 1, 2, 3$, associated with the vector fields X_i in (c).

(e) We introduce the function

$$f(x_1, .., x_4) = x_1x_4 - x_2x_3$$

We take as a known fact that the following four statements are equivalent (for each $k \neq 0$, and you may well assume $k = 1$):

(1) The 1-parameter groups $\{\theta_t^{(i)}\}$ transform the manifold M_k onto itself.

(2) The vector fields X_i annihilate the function f, that is, $X_i(f) = 0$.

(3) The 1-form df on $\mathbb{R}^4$ annihilates each X_i, that is, $< df, X_i >= 0$.

(4) The vector fields X_i are tangential to M_k, that is, are tangent vectors to M_k at each point on M_k.

Verify the statement (1), possibly by verifying one of the other three. If time permits, verify also the remaining statements.

(f) Let us write $M = M_1$. Use the results from (d) to show that the tangent bundle of M is trivial (parallelizable), that is, $TM \simeq M \times \mathbb{R}^3$.

(g) Show by direct calculation of Lie bracket that $[X_1, X_2] = X_3$.

Solutions

(a) Consider the map $f : \mathbb{R}^4 \to \mathbb{R}$, $(x_1, x_2, x_3, x_4) \to x_1x_4 - x_2x_3$. Its Jacobi matrix at a point $x = (x_1, .., x_4)$ is

$$(\frac{\partial f}{\partial x_1}, .., \frac{\partial f}{\partial x_4}) = (x_4, -x_3, -x_2, x_1)$$

and this has rank 1 unless all $x_i = 0$. Therefore, on $M_k, k \neq 0$, the rank is 1, and therefore k is a regular value of f and hence $M_k = f^{-1}(k)$ is a closed submanifold of dimension $4 - 1 = 3$. The manifold is not compact, since it is not a bounded subset of $\mathbb{R}^4$, namely the coordinates x_i can be arbitrarily large. (cf. Heine-Borel's theorem in point set topology).

(b) The linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$, $(x_1, x_2, x_3, x_4) \to (kx_1, kx_2, x_3, x_4)$, sends M_1 to M_k and certainly T yields a diffeomorphism between M_1 and M_k when $k \neq 0$. The union $\tilde{M}$ of all M_k for $k \neq 0$ is the complement of the closed set M_0, so $\tilde{M}$ is an open set.

(c) Let us set up the coefficients of the 3 vector fields X_i at a point $\mathbf{x}$ on $\tilde{M}$ as the column vectors of the following matrix

$$\begin{bmatrix} 0 & x_3 & x_1 \\ 0 & x_4 & x_2 \\ x_1 & 0 & -x_3 \\ x_2 & 0 & -x_4 \end{bmatrix}$$

Since $\mathbf{x}$ lies on some M_k and $k \neq 0$, we cannot have $x_1 = x_2 = 0$. Therefore, the 3×3-matrix given by the first three rows, or (if $x_1 = 0$) by removing the 3rd row, has non-zero determinant. Consequently, the rank of the whole matrix is always 3 whenever $\mathbf{x} \in \tilde{M}$.

(d) To integrate the vector field X_1 we must solve the dynamical system

$$\frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = 0, \frac{dx_3}{dt} = x_1, \frac{dx_4}{dt} = x_2$$

The first two equations yield $x_1 = c_1, x_2 = c_2$ where c_1 and c_2 are constants (depending on the initial value), and substituting for x_1 and x_2 into the last two equations yield

$$x_3 = tc_1 + c_3 \ , \ x_4 = tc_2 + c_4$$

where c_3 and c_4 are constants. Thus we can write in matrix notation

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \theta_t(\mathbf{x}_0)$$

where $\mathbf{x}_0 = \mathbf{x}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ is the initial value . Writing $\mathbf{x}(0)$ as $\mathbf{x}$, for simplicity, we see that the 1-parameter group $\{\theta_t\}$ is the group of linear transformations $\mathbf{x} \rightarrow \theta_t(\mathbf{x})$ given by the matrices $\theta_t = \theta_t^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{bmatrix}$. In a similar way we can easily integrate X_2 and X_3 to calculate the other two 1-parameter groups, briefly we get

$$\theta_t^{(2)} = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \theta_t^{(3)} = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

(e)

(1) Let us first show directly that each $\theta_t^{(i)}$ transforms M_k to itself. In each case, consider a point $\mathbf{x} = (x_1, ..x_4)$ in M_k and its image $\mathbf{x}'$ by the transformation $\theta_t^{(i)}$.We have

$$\text{case } i = 1 : \mathbf{x}' = (x_1', .., x_4') = (x_1, x_2, tx_1 + x_3, tx_2 + x_4)$$
$$\text{case } i = 2 : \mathbf{x}' = (x_1', .., x_4') = (x_1 + tx_3, x_2 + tx_4, x_3, x_4)$$
$$\text{case } i = 3 : \mathbf{x}' = (x_1', .., x_4') = (e^t x_1, e^t x_2, e^{-t} x_3, e^{-t} x_4)$$

In each case we find that $x_1'x_4' - x_2'x_3' = k$, that is, $\mathbf{x}'$ lies on M_k.

(2) The function $f(\mathbf{x}) = x_1x_4 - x_2x_3$ is annihilated by the vector fields X_i :

$$X_1(f) = x_1\frac{\partial f}{\partial x_3} + x_2\frac{\partial f}{\partial x_4} = x_1(-x_2) + x_2(x_1) = 0$$
$$X_2(f) = x_3\frac{\partial f}{\partial x_1} + x_4\frac{\partial f}{\partial x_2} = x_3(x_4) + x_4(-x_3) = 0$$
$$X_3(f) = x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} - x_3\frac{\partial f}{\partial x_3} - x_4\frac{\partial f}{\partial x_4} = x_1x_4 + x_2(-x_{3)} - x_3(-x_2) - x_4x_1 = 0$$

(3) $df = x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4$:

$< df, X_1) =< x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4, x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4} >= 0 - 0 - x_2x_1 + x_1x_2 = 0$

$< df, X_2) =< x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4, x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} >= x_4x_3 - x_3x_4 + 0 + 0 = 0$

$< df, X_3) =< x_4dx_1 - x_3dx_2 - x_2dx_3 + x_1dx_4, x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} >= x_4x_1 - x_3x_2 - x_2(-x_3) + x_1(-x_4) = 0$

(4) By (1), the integral curves $t \to \gamma(t) = \theta_t^{(i)}(\mathbf{x})$ lie on M_k whenever the point $\mathbf{x}$ lies on M_k. Therefore their velocity vectors are tangent vector to M_k. On the other hand, there is a simple way to describe the tangent space $T_{\mathbf{x}}M_k$, as a 3-dimensional subspace of $\mathbb{R}^4 = T_{\mathbf{x}}\mathbb{R}^4$. Namely, since M_k is the level surface $f^{-1}(k)$ of the function f, it is not difficult to see that the (usual) gradient vector

$$\nabla f = x_4\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}$$

is perpendicular to M_k, that is, the orthogonal complement of the vector space $T_{\mathbf{x}}M_k$ in $\mathbb{R}^4$ is the line spanned by ∇f. Now, by calculating the usual (Euclidean) inner product of vectors in $\mathbb{R}^4$ we find easily

$$\nabla f \cdot X_1 = \nabla f \cdot X_2 = \nabla f \cdot X_3 = 0$$

and hence X_1, X_2, X_3 lie in the tangent space of M_k.

(f) Let us construct a map $\Phi : M \times \mathbb{R}^3 \to TM$ by setting

$$\Phi(\mathbf{x}, (t_1, t_2, t_3)) = t_1X_1(\mathbf{x}) + t_2X_2(\mathbf{x}) + t_3X_3(\mathbf{x})$$

This is smooth map, and for fixed $\mathbf{x}$ it is a linear isomorphism between $\mathbb{R}^3 = \{\mathbf{x}\} \times \mathbb{R}^3$ and $T_{\mathbf{x}}M$. We also claim the inverse map Φ^{-1} is smooth, so Φ is a diffeomorphism. All this is equivalent to having a smooth 3-frame on M, such as (X_1, X_2, X_3). The student is not expected to do more calculations in order to verify the smoothness of Φ^{-1}.

Remark To calculate Φ^{-1} is the same as determining the coefficients t_i of any vector in TM with respect to the frame. One way to do this would be to introduce a Riemannian metric on TM and apply the Gram-Smith procedure to replace our frame by an orthonormal frame (E_1, E_2, E_3). Then the expressions for t_1, t_2, t_3 would follow easily.

(g) We calculate

$$\begin{aligned}
[X_1, X_2] &= [x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4},\ x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2}] \\
&= [x_1\frac{\partial}{\partial x_3}, x_3\frac{\partial}{\partial x_1}] + [x_1\frac{\partial}{\partial x_3}, x_4\frac{\partial}{\partial x_2}] + [x_2\frac{\partial}{\partial x_4}, x_3\frac{\partial}{\partial x_1}] + [x_2\frac{\partial}{\partial x_4}, x_4\frac{\partial}{\partial x_2}] \\
&= (x_1\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_3}) + 0 + 0 + (x_2\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_4}) = X_3
\end{aligned}$$

Problem 3

In this exercise we shall construct a smooth embedding of the real projective plane $\mathbb{R}P^2$ into $\mathbb{R}^4$ by means of the function $F : \mathbb{R}^3 \to \mathbb{R}^4$, where

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz)$$

(a) Explain why we have a well defined smooth mapping $\bar{F} : \mathbb{R}P^2 \to \mathbb{R}^4$ such that $F = \bar{F} \circ \pi$, that is, the following diagram commutes :

$$\begin{array}{ccc} S^2 & \longrightarrow^F & \mathbb{R}^4 \\ \pi \downarrow & \nearrow & \\ \mathbb{R}P^2 & & \end{array}$$

Here S^2 is the unit sphere $x^2 + y^2 + z^2 = 1$ of $\mathbb{R}^3$ and we assume it is known that the quotient map $\pi : (x, y, z) \to [x, y, z]$ is a local diffeomorphism.

Now, explain briefly why it follows from (b) and (c) below that $\bar{F}$ must be an embedding.

(b) F as a function on S^2 has rank 2 at each point. Show this.

(c) F takes the same value only at antipodal points $\pm p$ on S^2. Show this.

(Hint: Here it can be useful to consider first the function $(x, y) \to (x^2 - y^2, xy)$, and perhaps make use of spherical coordinates, $x = \sin\varphi\cos\theta, y = \sin\varphi\sin\theta, z = \cos\varphi$).

Solutions

(a) The map $\bar{F}$ exists because if $\pi(p_1) = \pi(p_2)$, then $p_1 = \pm p_2$ and hence (by inspection) $F(p_1) = F(p_2)$. Thus we can define $\bar{F}$ by setting $\bar{F}(\pi(p)) = F(p)$. The function $\bar{F}$ will be continuous because (by definition) $\mathbb{R}P^2$ has the quotient topology of S^2 via π (or equivalently, as a quotient space of $\mathbb{R}^3 - \{0\}$). We are told that π is a local diffeomorphism, namely it is a smooth map which is locally invertible.

Now, if (b) holds, then F has rank 2, and locally we can write $\bar{F} = F \circ \pi^{-1}$, which also has rank 2. Finally, if (c) holds, then $\bar{F}$ is also injective, and hence it is an injective immersion. Moreover, $\mathbb{R}P^2$ is compact since it is the image of S^2 by the continuous map π. Therefore, $\bar{F}$ is a homeomorhism onto its image $\bar{F}(\mathbb{R}P^2)$ in $\mathbb{R}^4$, and $\bar{F}(\mathbb{R}P^2)$ is also a submanifold of the latter.

(b) To show that F has rank 2, let us first calculate the Jacobi matrix of F as a function on $\mathbb{R}^3$:

$$DF = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$$

If $x \neq 0$ or $y \neq 0$, then the matrix has rank 3, so it is an injective linear map $\mathbb{R}^3 \to \mathbb{R}^4$. In particular, restricted to the tangent plane of S^2 at $p = (x, y, z) \in$

S^2 it is still injective. Hence, the map has rank 2 at p regarded as a map of S^2. Finally, if $x = y = 0$, then $p = (0, 0, \pm 1)$ and the tangent plane of S^2 at p is the xy-plane in $\mathbb{R}^3$, whereas the linear map $dF : \mathbb{R}^3 \to \mathbb{R}^4$ has the matrix

$$DF = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{pmatrix}$$

We see that the xy-plane (i.e. $z = 0$) in $\mathbb{R}^3$ is mapped injectively into $\mathbb{R}^4$, so the rank of F is again 2, viewed as a map of S^2.

(c) We have to show that if $F(p) = F(q)$ for points p, q on S^2, then either $p = q$ or $p = -q$.

Consider first the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x^2 - y^2, xy)$. Let us try to invert this map, namely by solving the system

$$x^2 - y^2 = a, xy = b$$

for (x, y) in terms of (a, b). We calculate

$$\begin{aligned} a^2 + 4b^2 &= x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2 \\ x^2 + y^2 &= \sqrt{a^2 + 4b^2} \\ x^2 &= \frac{1}{2}(a + \sqrt{a^2 + 4b^2}),\ y^2 = \frac{1}{2}(-a + \sqrt{a^2 + 4b^2}) \end{aligned}$$

Thus $\pm x$ and $\pm y$ are uniquely determined by a, b, and since xy is already determined by b, only the two pairs $\pm(x, y)$ are mapped to a given pair (a, b).

Next, we consider the map $F : (x, y, z) \to (x^2 - y^2, xy, xz, yz)$ for points $(x, y, z) \in S^2$. If (x_1, y_1, z_1) and (x_2, y_2, z_2) are mapped to the same point, then we already know $(x_1, y_1) = (x_2, y_2)$ or $(-x_2, -y_2)$. It is now easy to check that $z_1 = \pm z_2$ and with sign $\pm$ so that $(x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$, that is, the two points on S^2 are antipodal.