Øving 12, TMA 4190, April 2014 !

Kladd !

9-4 $n \geq 1$, $S^{2n-1} \subset \mathbb{C}^n$

$$\theta(t,z) = e^{it} z = \theta_t(z) = e^{it} \cdot z \qquad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$|z_i|^2 = 1$$

flow.

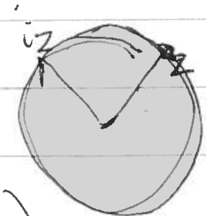
$$\begin{array}{ccc} \mathbb{R} \times S^{2n-1} & \xrightarrow{\theta} & S^{2n-1} \\ \cap & \cap & \cap \\ \mathbb{R} \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \end{array}$$

infinitesimal generator X $\qquad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

vektfelt på $\mathbb{C}^n$ $\qquad X_z = \frac{d}{dt}\Big|_{t=0} \theta(t,z)$

$$= \frac{d}{dt}\Big|_{t=0} e^{it} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = i e^{it}\Big|_{t=0} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = i \begin{pmatrix} z_1 \\ z_2 \\ \vdots \end{pmatrix}$$

$$X_z = iz = \begin{pmatrix} iz_1 \\ iz_2 \\ \vdots \\ iz_n \end{pmatrix}$$



$$z = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \longleftrightarrow \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix} \longrightarrow X_z = \begin{pmatrix} ix_1 - y_1 \\ ix_2 - y_2 \\ \vdots \\ ix_n - iy_n \end{pmatrix} \longleftrightarrow \begin{pmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \\ \vdots \\ -y_n \\ x_n \end{pmatrix}$$

Så $X_z \perp z$, $\quad T_z S^{2n-1} = (z)^{\perp}$ i $\mathbb{R}^{2n}$

Så X_z er tangentfelt til S^{2n-1} når $z \in S^{2n-1}$.

9-18

(a) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

$\frac{dx}{dt} = x \,,\ \frac{dy}{dt} = -y$

$x = x_0 e^{t} \,,\ y = y_0 e^{-t}$

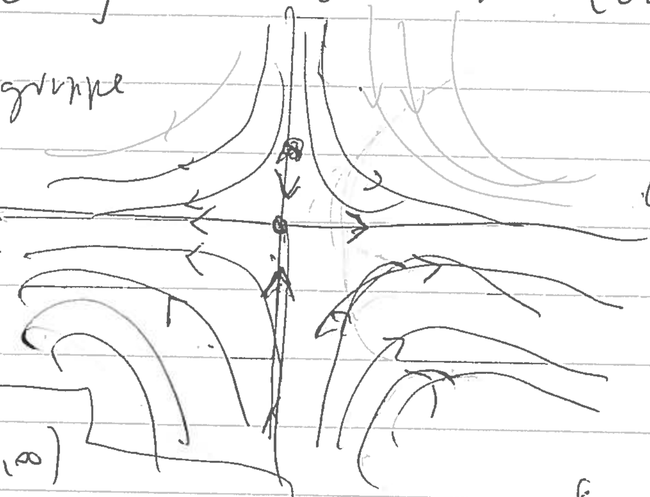
$\theta_t = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

$xy = x_0 y_0 = \text{constant}$

1-param gruppe

$\simeq \{e^{t} : t \in \mathbb{R}\}$

$\simeq (\mathbb{R}, +)$



integralkurver er hyperbler

$\mathbb{R} \longrightarrow \mathbb{R}^{+} = (0, \infty)$

$t \longrightarrow e^{t}$ group homo.

med log som inverse

$\{\theta_t\} \simeq (\mathbb{R}, +)$

(b) $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$

$\frac{dx}{dt} = y \,,\ \frac{dy}{dt} = x$

$x'' = x$ osv

Over ~~til~~ side 3

9-18

$$Y = x\frac{d}{dy} + y\frac{d}{dx} = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x$$

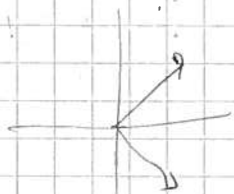
$$\frac{d^2x}{dt^2} = x \qquad x'' = x \Rightarrow x = ae^{t} + be^{-t}$$

$$\cancel{x' = ae^{t} - be^{-t}}$$
$$\cancel{x'' = ae^{t} + be^{-t}}$$

$$\boxed{\begin{aligned} x &= ae^{t} + be^{-t} \\ y &= ae^{t} - be^{-t} \end{aligned}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left.\begin{aligned} x_0 &= a + b \\ y_0 &= a - b \end{aligned}\right\} \qquad a = \frac{x_0 + y_0}{2}, \quad b = \frac{x_0 - y_0}{2}$$



$$x = \frac{x_0 + y_0}{2}e^{t} + \frac{x_0 - y_0}{2}e^{-t} = \left(\frac{e^{t} + e^{-t}}{2}\right)x_0 + \left(\frac{e^{t} - e^{-t}}{2}\right)y_0$$

$$y = \frac{x_0 + y_0}{2}e^{t} - \left(\frac{x_0 - y_0}{2}\right)e^{-t} = \left(\frac{e^{t} - e^{-t}}{2}\right)x_0 + \left(\frac{e^{t} + e^{-t}}{2}\right)y_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh t\, x_0 + \sinh t\, y_0 \\ \sinh t\, x_0 + \cosh t\, y_0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\det = 1$

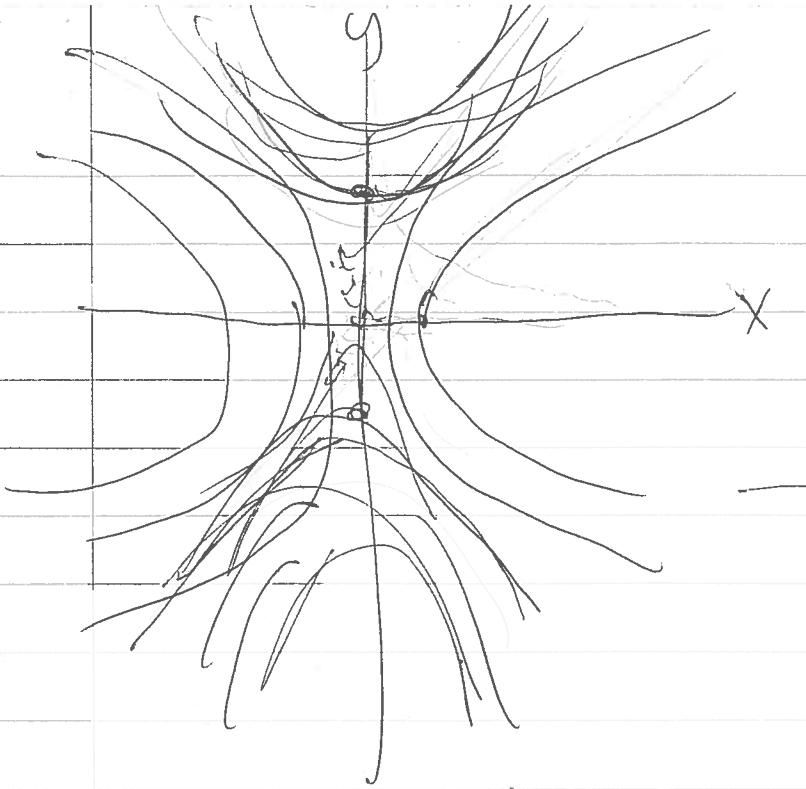
$SL(2,\mathbb{R})$

$$\Psi_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

def. for alle t

$$\{\Psi_t\} \simeq \mathbb{R} \cong SO(1,1)$$

hyperbolsk rotasjon!



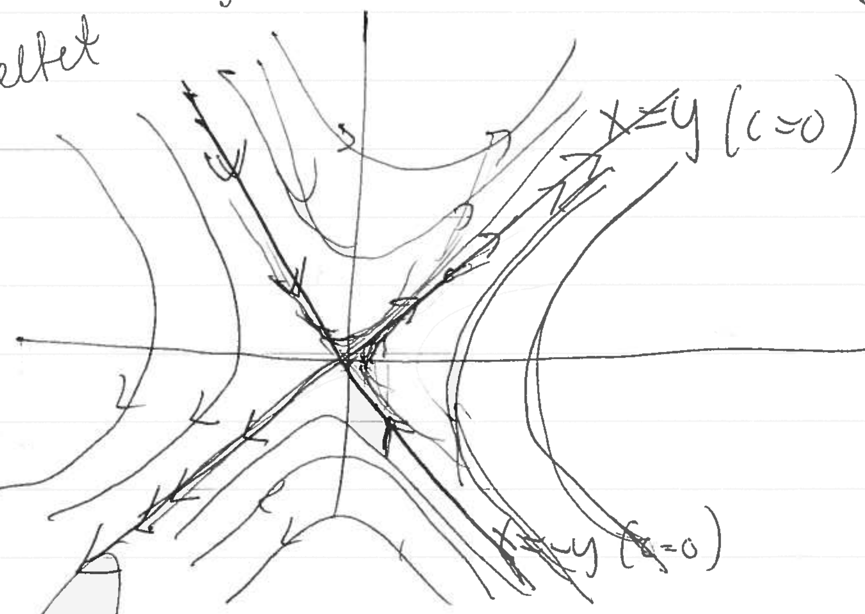
$x^2 - y^2 = x_0^2 - y_0^2 = c^2$

const

hyperblar er integralkurvene

$x = y \quad c = 0$

Y feltet



Roter vektorfeltet X (eller Y) $\frac{\pi}{4}$ vinkel om origo

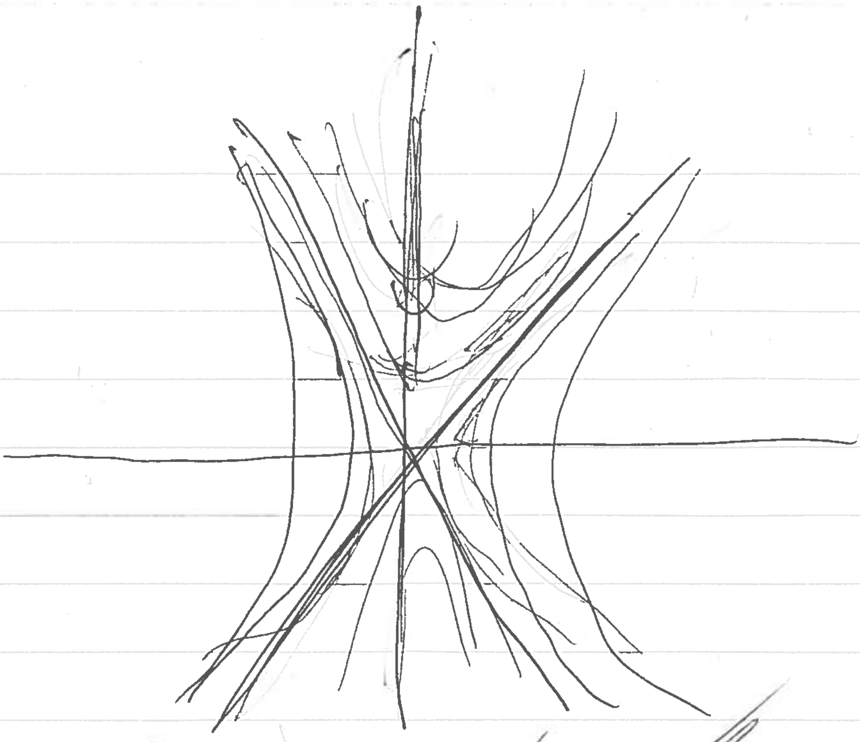
$$\begin{tikzcd} T\mathbb{R}^2 \arrow[r, "d\rho"] & T\mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "Y"] \arrow[r, "\rho"] & \mathbb{R}^2 \arrow[u, "X"'] \end{tikzcd}$$

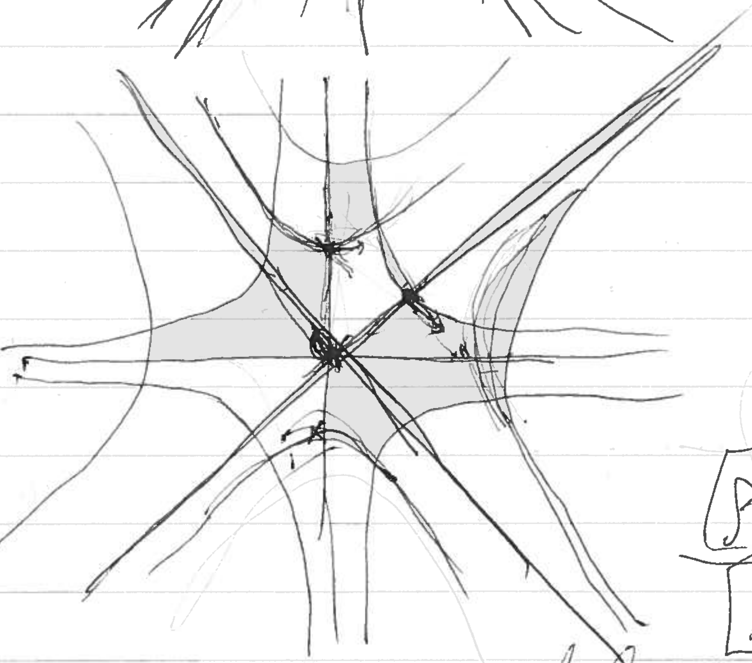
Er $\rho_* Y = X$?

ρ = rotasjon $\pi/4$

$$\rho\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\sqrt{2} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$(x,y;\, y,x) \xrightarrow{d\rho} (x',y';\, y',x')$$





~~v~~ inkelrett
på hverandre

$\boxed{\rho = 180^\circ \text{ rotasjon}}$

$\boxed{\rho_* X = X}$ $\rho_* Y = Y$

$X_{(x,y)} = (x,-y)$

$Y_{(x,y)} = (y,x)$

$\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

~~ρ = rotasjon~~ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{array}{ccc} T\mathbb{R}^2 & \xrightarrow{d\rho = (\rho,\rho)} & T\mathbb{R}^2 \\ \uparrow & & \uparrow \\ \mathbb{R}^2 & \xrightarrow{\rho} & \mathbb{R}^2 \end{array}$$

$$\begin{array}{ccc} (x,y,x,-y) & \xrightarrow{d\rho} & (y,-x,y,x) \\ \uparrow X & & \uparrow X \\ (x,y) & \xrightarrow{\rho} & (y,-x) \end{array}$$

$X_{(-y,x)} = (-y,-x)$

rot 90° $(-y,x)$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$

La $\phi = 90^\circ$ rotasjon $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ på $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\begin{array}{ccc} (x,y;x,-y) & \xrightarrow{d\phi} & (-y,x;y,x) \\ \uparrow X & & \uparrow \phi_* X \\ (x,y)\,\mathbb{R}^2 & \xrightarrow{\phi_*} & (-y,x) \end{array}$$

$$\phi_*(X)_{(x,y)} = (-x,y)$$

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \longrightarrow = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$= -X$$

Analogt $\phi_* Y = -Y$

La $\psi = 45^\circ$ rot $= \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$ pos. retn.

$$\begin{array}{ccc} (x,y,x,-y) & \xrightarrow{d\psi} & (x',y';x\tfrac{1}{2}\sqrt{2}+y\tfrac{1}{2}\sqrt{2},\ x\tfrac{1}{2}\sqrt{2}-y\tfrac{1}{2}\sqrt{2}) \\ \uparrow X & & \uparrow ? \\ (x,y) & \xrightarrow{\psi} & (x',y') = (x\tfrac{1}{2}\sqrt{2}-y\tfrac{1}{2}\sqrt{2},\ x\tfrac{1}{2}\sqrt{2}+y\tfrac{1}{2}\sqrt{2}) \end{array}$$

$$= \tfrac{1}{2}\sqrt{2}(x-y,\ x+y,\ x+y,\ x-y$$

$$= \tfrac{1}{2}\sqrt{2}(x-y,\ x+y)$$

$$(\psi_* X)_{(x,y)} = (y,x) = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y$$

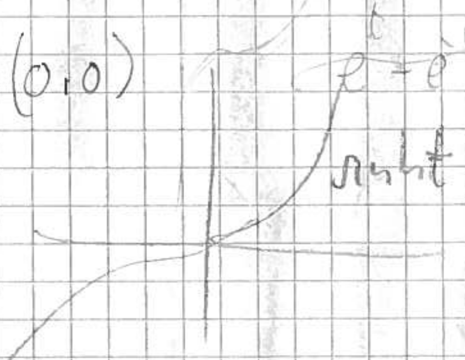
$$\Psi_t \circ \Theta_s = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} = \begin{pmatrix} \cosh t \, e^{s} & \sinh t \, e^{-s} \\ \sinh t \, e^{s} & \cosh t \, e^{-s} \end{pmatrix}$$

$$\underset{\Theta_s}{\begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix}} \underset{\Psi_t}{\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}} = \begin{pmatrix} \cosh t \, e^{s} & \sinh t \, e^{s} \\ \sinh t \, e^{-s} & \cosh t \, e^{-s} \end{pmatrix}$$

~~så velg s,t slik at~~

~~men e^{-s}~~ for $s,t \neq (0,0)$

vil vi ha



$t \neq 0 \Rightarrow \sinh t \neq 0$

$\sinh t \, e^{-s} \neq \sinh t \, e^{s}$

$s \neq 0$

så $\Psi_t \Theta_s \neq \Theta_s \Psi_t$ generelt

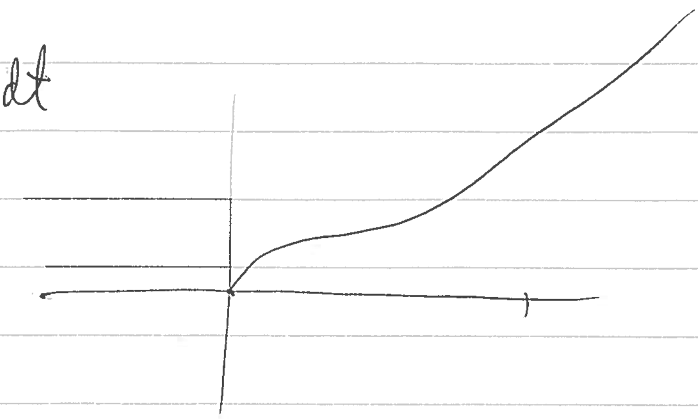
13-5 Parametrisering med linelengd

$\gamma(t) : \mathbb{R} \longrightarrow M$

$|\gamma'(t)|_g \neq 0$

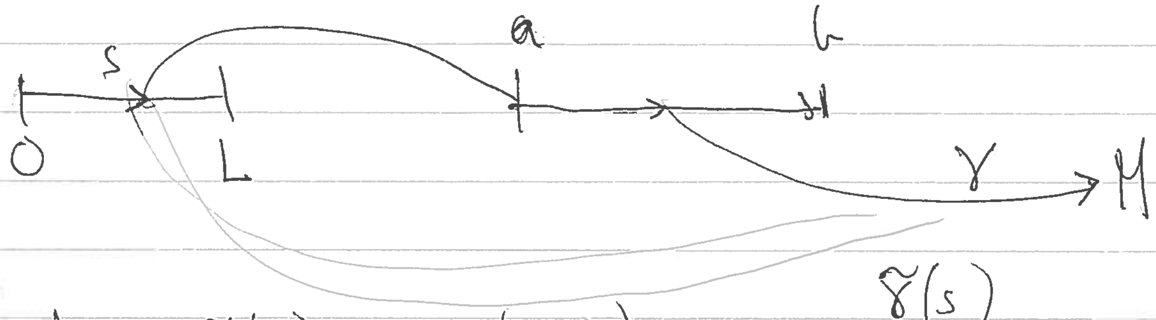
set $\cancel{s(t)=} \; ds = \int_0^t |\gamma'(t)| \, dt$

$s(t) = \int_0^t |\gamma'(t)|_g \, dt$



$\frac{ds}{dt} = |\gamma'(t)| > 0$

$s \longleftrightarrow t$ glatt og invertibel



$d\tilde{\gamma}$ $\quad \tilde{\gamma}(s) = \gamma(t(s))$

$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt}(t(s)) \cdot t'(s)$ $\qquad \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$

$\left|\frac{d\tilde{\gamma}}{ds}\right| = \left|\frac{d\gamma}{dt}(\,)\right|_g \frac{dt}{ds} = \frac{ds}{dt} \cdot \frac{dt}{ds} = 1$

13-6 $M \overset{\text{diffeo}}{\approx} S^1$ eller $\mathbb{R}$ (hvis sammenhengende

(a) $M = \mathbb{R}$, (x = koord) $\quad ds^2 = g(x)\,dx^2 \quad$ ~~d~~ $g(x) > 0$ overalt

skriver $k(x) = \sqrt{g(x)}$, $\quad ds^2 = k^2(x)\,dx^2 = (k(x)\,dx)^2 = dy^2$

$dy = k(x)dx$

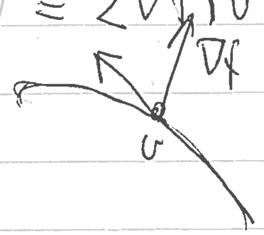
$\int dy = y = \int k(x)dx$ ny koord. funksjon på $\mathbb{R}$

Lokalt er dette ok for alle 1-dim mfd., så også

S^1 ...

13-21 $v(f) = df(v) = \langle \nabla f, v \rangle_g$ er størst når ∇f og v peker i samme retn, ...

(i) $v \in T_pM$



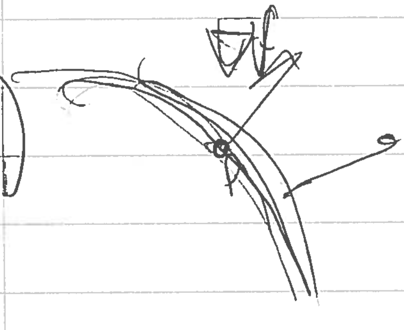
$v = \boxed{\frac{\nabla f}{\|\nabla f\|}}$ gir størst retn. derivering $v(f)$

$\nabla f = (\nabla f) \cdot$

$v(f) = \langle \nabla f, v \rangle$

$= \langle \nabla f, \frac{\nabla f}{\|\nabla f\|} \rangle = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$

(ii)



La $M = f^{-1}(c)$,

og la $\gamma(t)$ være kurve på M gjennom p ($t=0$)

$$\langle \frac{d\gamma}{dt}(p), \nabla f|_p \rangle_g = \frac{d\gamma}{dt}\Big|_p (f) = \frac{d}{dt}\Big|_{t=0} \left(f(\gamma(t)) \right) = \frac{dc}{dt}\Big|_{t=0} = 0$$