Øving 11, TMA4190, April 2014

Lee 8-16 (a) Finn $[X,Y]$, når $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$, $Y = \frac{\partial}{\partial y}$ i $\mathbb{R}^3(x,y,z)$.

$$[X,Y] = [y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}, \frac{\partial}{\partial y}] = [y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}] - [2xy^2\frac{\partial}{\partial y}, \frac{\partial}{\partial y}]$$

$$= -\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial z}\right) + \frac{\partial}{\partial y}\left(2xy^2\frac{\partial}{\partial y}\right) - 2xy^2\frac{\partial^2}{\partial y} = -\frac{\partial}{\partial z} + 0$$

$$+ 4xy\frac{\partial}{\partial y} + 0 = -\frac{\partial}{\partial z} + 4xy\frac{\partial}{\partial y}$$

(b) $$[X,Y] = [x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}] = [x\frac{\partial}{\partial y}, y\frac{\partial}{\partial z}] - [y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z}]$$

$$- [x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}] + [y\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}] = x\frac{\partial}{\partial z} - 0 + 0 - 0 - z\frac{\partial}{\partial x}$$

$$= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

(c) $$[X,Y] = [x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}] = [x\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}] - [y\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}]$$

$$+ [x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}] - [y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}] = 0 + 0 + 2[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}]$$

$$= 2\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

Lee 8-19 $\mathbb{R}^3$, $u \times v \overset{\text{def}}{=} [u,v]$, Lie-algebra $\overset{?}{}$

Pga kjende reglar for kryss-produktet har vi

1) $[u,v] = -[v,u]$,

2) bilinear produktregel,

Men: kva med Jacobi-identiteten?

La oss bruke at $(u \times v) \times w = -(v \cdot w)u + (u \cdot w)v$!
Det gir

$$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$$

ved direkte utrekning. OK

Lee 8-20 $\quad X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \; Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \; Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

I oppg. 8–16(b) viste vi at $[Z, X] = -Y$

$A = \text{lin}\{X, Y, Z\} < \mathfrak{X}(\mathbb{R}^3)$ er eit 3-dim underrom, altså isomorf med $\mathbb{R}^3$. Påstanden er at A er også isomorf med $\mathbb{R}^3$ som Lie-algebr (med kryss-produktet, dvs. $u \times v \overset{\text{def}}{=} [u, v]$)

Skriv $E_1 = X, \; E_2 = -Y, \; E_3 = Z$

Vi finn at

$$[E_1, E_2] = -[X, Y] = -\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]$$
$$= -\left\{y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right\} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = Z = E_3$$

Sameleis, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$

I $\mathbb{R}^3$ har vi $e_1 \overset{\text{def}}{\underset{=[e_i,e_j]}{\times}} e_2 = e_3, \; e_2 \times e_3 = e_1, \; e_3 \times e_1 = e_2$

Definer lineær isomorfi

$$\mathbb{R}^3 \xrightarrow{\phi} A, \quad \left.\begin{aligned} e_1 &\to E_1 = \phi(e_1) \\ e_2 &\to E_2 = \phi(e_2) \\ e_3 &\to E_3 = \phi(e_3) \end{aligned}\right\}$$

Då er $[\phi(e_i), \phi(e_j)] = \phi([e_i, e_j]), \; \forall i, j$
Så ved linearitet til $[\, , \,]$ ser vi ϕ blir ein Lie-algebra isomorfi, dvs. $[\phi(u), \phi(v)] = \phi([u, v]), \; \forall u, v$