Norges teknisk-naturvitenskapelige universitet
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SOLUTIONS

Eksamen i TMA4190 Mangfoldigheter

Onsdag 4 juni, 2013. Tid : 09.00–13.00

Oppgave 1

a) La $U \subset \mathbb{R}^n$ være enhetsdisken $|\mathbf{x}| < 1$. Vis at avbildningen

$$F : U \to \mathbb{R}^n, \quad F(\mathbf{x}) = \frac{\mathbf{x}}{1 - |\mathbf{x}|^2}$$

er en diffeomorfi.

Solution: It is not difficult to calculate the inverse F^{-1} directly. Look for example at the simplest case of $n = 1$, and replace x^2 by $|\mathbf{x}|^2$ when $n > 1$. Then we get

$$F^{-1} : \mathbb{R}^n \to U, \ \mathbf{y} \to \frac{2\mathbf{y}}{1 + \sqrt{1 + 4|y|^2}}$$

b) Vis at ellipsoiden gitt ved likningen

$$4x^2 + 2y^2 + 5z^2 = 2$$

er en lukket imbeddet undermangfoldighet av $\mathbb{R}^3$.

Solution: Consider the function $f(x, y, z) = 4x^2 + 2y^2 + 5z^2$ from $\mathbb{R}^3$ to $\mathbb{R}$. The ellipsoid is the level surface $E = f^{-1}(2)$. According to a basic theorem that we constantly use, all we need to do is to verify that 2 is a regular value of f, namely that all the points on E are regular points for f. Calculation of the gradient of f yields

$$\nabla f = (8x, 4y, 10z),$$

and this vanishes only at the origin. So all points on E are regular for f.

c) La M være delmengda i $\mathbb{R}^3$ definert ved ligningen

$$x^3 + y^3 + z^3 = 2xyz + 1$$

Vis at M har en glatt struktur som en 2-dimensjonal mangfoldighet.

Solution: This is similar to case b), setting $f(x, y, z) = x^3 + y^3 + z^3 - 2xyz$ and $M = f^{-1}(1)$.Then the gradient

$$\nabla f = (3x^2 - 2yz, 3y^2 - 2xz, 3z^2 - 2xy)$$

vanishes if and only if $3x^2 = 2yz$,$3y^2 = 2xz$, $3z^2 = 2xy$.This implies $27x^2y^2z^2 = 8x^2y^2z^2$ and hence $xyz = 0$. But now the three previous equations together imply $(x, y, z) = (0, 0, 0)$. However, the origin $(0, 0, 0)$ lies outside M, so all points on M are regular for f.

Oppgave 2

Vi ser på følgende tre vektorfelt på $\mathbb{R}^3$:

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (1 + x^2)\frac{\partial}{\partial z}$$

a) Vis at disse vektorfeltene danner en basis for modulen av glatte vektorfelt på $\mathbb{R}^3$, nemlig at for ethvert vektorfelt X så finnes glatte reelle funksjoner f, g, h på $\mathbb{R}^3$ slik at

$$X = fE_1 + gE_2 + hE_3 \tag{1}$$

Solution: We need to show that at each point $p = (x, y, z)$ the three vector fields are linearly independent and hence constitute a basis of the tangent space at p. This follows from

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 + x^2 \end{pmatrix} = 1 + x^2 \neq 0$$

Therefore, any other vector X_p at p is a linear combination $X_p = fE_1|_p + gE_2|_p + hE_3|_p$ for suitable numbers f, g, h. But this holds at each point p so we can regard f, g, h as functions of p and thus obtain the identity (**??**) for vector fields.

b) Bestem de duale 1-formene ω_1,ω_2,ω_3 til E_1, E_2, E_3, dvs. slik at $\omega_i(E_j) = \delta_{ij} = \left\{\begin{smallmatrix} 1\ ,i=j \\ 0,i\neq j \end{smallmatrix}\right.$, uttrykt ved dx, dy, dz.

Solution: Consider the 1-form $\omega_1 = Adx + Bdy + Cdz$, where we determine the coefficient functions A, B, C as the solution of a linear system of equations:

$$\begin{aligned} 1 &= \omega_1(E_1) = (Adx + Bdy + Cdz)(E_1) = A \\ 0 &= \omega_1(E_2) = A + B \\ 0 &= \omega_1(E_3) = A + B + (1 + x^2)C \end{aligned}$$

The solution is $A = 1, B = -1, C = 0$, hence $\omega_1 = dx - dy$. Similarly, we obtain

$$\omega_2 = dy - \frac{1}{1 + x^2}dz, \;\; \omega_3 = \frac{1}{1 + x^2}dz$$

c) Vis først at

$$[E_1, E_2] = 0, [E_1, E_3] = [E_2, E_3] = 2x\frac{\partial}{\partial z}$$

og verifiser deretter Jacobi- identiteten

$$[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2] = 0$$

Solution : Note first that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = [\frac{\partial}{\partial y}, \frac{\partial}{\partial z}] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial z}] = 0$. Then

$$[E_1, E_3] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (1+x^2)\frac{\partial}{\partial z}] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial x}] + [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] + [\frac{\partial}{\partial x}, (1+x^2)\frac{\partial}{\partial z}] =$$
$$= 0 + 0 + \frac{\partial}{\partial x}(1+x^2)\frac{\partial}{\partial z} + (1+x^2)\frac{\partial}{\partial x}\frac{\partial}{\partial z} - (1+x^2)\frac{\partial}{\partial z}\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial z}$$

and similarly we find $[E_2, E_3] = 2x\frac{\partial}{\partial z}$.. Then

$$[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2]$$
$$= 0 + [2x\frac{\partial}{\partial z}, \frac{\partial}{\partial x}] - [2x\frac{\partial}{\partial z}, \frac{\partial}{\partial x} + \frac{\partial}{\partial y}] = [2x\frac{\partial}{\partial z}, \frac{\partial}{\partial x}] - [2x\frac{\partial}{\partial z}, \frac{\partial}{\partial x}] - [2x\frac{\partial}{\partial z}, \frac{\partial}{\partial y}]$$
$$= -2x\frac{\partial}{\partial z}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}(2x\frac{\partial}{\partial z}) = -2x\frac{\partial}{\partial z}\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial y}\frac{\partial}{\partial z} = 0$$

Oppgave 3

a) Vi ser på en glatt avbildning $f : M \to N$ mellom to glatte mangfoldigheter. Forklar kort hva det vil si at f er en

1) immersjon, 2) submersjon, 3) imbedding

og gi eksempel på en injektiv immersjon som ikke er en imbedding.

Solution : Consider the induced linear map at a point $p \in M$, $f_*|_p : T_pM \to T_{f(p)}N$.

Immersion means that $f_*|_p$ is injective at each point, submersion means that it is surjective. Imbedding means that the image $f(M)$ is a submanifold of N, and the map $f : M \to f(M)$ is a diffeomorphism. An injective immersion fails to be a diffeomorphism when the map $f : M \to f(M)$ is not a homeomorphism, when $f(M)$ has the subspace topology from N. In fact, by definition, a submanifold must have the subspace topology.

As an example of an injective immersion which is not an imbedding, consider a figure 8 curve in the plane with the selfcrossing at the origin $(0,0)$ and the loops in the halfspaces $x > 0$ and $x < 0$. Parametrize the curve $\sigma : \mathbb{R} \to \mathbb{R}^2$ so that $\sigma'(t) \neq 0$ for all t, $\sigma(0) = (0,0) = \lim \sigma(t)$ as $t \to \pm\infty$. Clearly, with the subspace topology from $\mathbb{R}^2$ the figure will not be homeomorphic to $\mathbb{R}$.

b) Avbildningen $f : \mathbb{R}^3 \supset S^2 \to \mathbb{R}^4$ er definert ved

$$f : (x, y, z) \to (x^2 - y^2, xy, xz, yz)$$

Vis at f induserer en imbedding $\bar{f} : \mathbb{R}P^2 \to \mathbb{R}^4$, hvor $\mathbb{R}P^2$ er det reelle projektive plan, oppfattet her som et kvotientrom av 2-sfæren S^2, med den vanlige glatte strukturen.

Solution : We consider the diagram

$$\begin{array}{ccc} S^2 & \longrightarrow^f & \mathbb{R}^4 \\ \pi \downarrow & \searrow & \cup \\ \mathbb{R}P^2 & \longrightarrow^{\bar{f}} & M \end{array}$$

where M is the image set $f(S^2)$. As usual, π is the quotient map when antipodal points on S^2 are identified. Antipodal points are mapped to the same point in $\mathbb{R}^4$ by f , so this induces the map $\bar{f}$ as indicated. But also we can show that $f(u) = f(v)$ if and only if $u = \pm v$, so $\bar{f}$ is injective. Since $\mathbb{R}P^2$ is compact and $\bar{f}$ is a bijective and continuous map, $\bar{f}$ will be a homeomorphism.

Finally, it remains to show that $\bar{f}$ is smooth and is an immersion. But the differentiable structure of $\mathbb{R}P^2$ is inherited from S^2, so that the 2-1 map π is a local diffeomorphism. Smoothness is a local property of a function, so f is smooth if and only if $\bar{f}$ is smooth. The rank of a function at a point is also a local property, so we need to show that f (and hence also $\bar{f}$) has rank 2 everywhere on the 2-sphere.

The Jacobi matrix of f as a function on $\mathbb{R}^3$ is

$$Df = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$$

Consider first a point $p = (0, 0, \pm 1)$ on S^2, so that $x = y = 0$. The matrix has rank 2, so some 1-dimensional subspace of $T_p\,\mathbb{R}^3 = \mathbb{R}^3$ is mapped to 0. But one checks easily that the tangent vectors

$(a, b, 0)$ of S^2 at p are mapped injectively, so the rank of $f|_{S^2}$ is 2 at p. Next, if $p = (x, y, z)$, $z \neq 1$, $x \neq 0$, then the upper 3×3- determinant of Df is nonzero, so the rank of Df is 3, so again the tangent space of S^2 is mapped injectively. Finally, if $p = (0, y, z)$, $z \neq 1$, $y \neq 0$, is a point on S^2 , then the rank of Df is 3, so $f|_{S^2}$ has again maximal rank 2 at p. So, with some case by case analysis we can show that f restricted to S^2 is an immersion, hence also $\bar{f}$.

Question !!! Do you think it would be possible to find some smooth imbedding $\mathbb{R}P^2 \to \mathbb{R}^3$, somehow? Well, how would the projective plane, which is not orientable, look like in our 3-space? The answer is no, and this is an important fact in topology.

Oppgave 4

a) Integrer vektorfeltet på $\mathbb{R}^2$ definert ved

$$\mathbf{F} = x^2 \frac{\partial}{\partial x} + (2 - y)\frac{\partial}{\partial y}$$

og angi den generelle løsningen som en (lokal eller global) flow eller 1-parameter gruppe. For integralkurven som ved $t = 0$ er i punktet $(1, 1)$, for hvilke vedier av t er denne definert?

Solution. We integrate the system

$$\frac{dx}{dt} = x^2, \quad \frac{dy}{dt} = 2 - y$$

and find the solutions

$$x = \frac{1}{A-t}, \quad y = 2 + Be^{-t}$$

eor

$$(x(t), y(t)) = (\frac{x_0}{1 - x_0 t}, 2 + (y_0 - 2)e^{-t})$$

For $(x_0, y_0) = (1, 1)$ we get the solution $(x(t), y(t)) = (\frac{1}{1-t}, e^{-t})$, which is defined for $t \in (-\infty, 1)$.

b) Vi definerer en 1-parameter gruppe $\{\varphi_t\}$ på $\mathbb{R}^3$ ved at

$$\varphi_t(x, y, z) = (\cos t\ x - \sin t\ z, y, \sin t\ x + \cos t\ z)$$

Bestem det induserte vektorfeltet $\mathbf{V}$ på $\mathbb{R}^3$, og gjør rede for hvorfor $\mathbf{V}$ er tangentiell til sfæren $S^2 : x^2 + y^2 + z^2 = 1$. Forsøk også å uttrykke vektorfeltet $\mathbf{V}$ på S^2 som en lineær kombinasjon av $\frac{\partial}{\partial\theta}$ og $\frac{\partial}{\partial\varphi}$, hvor (θ, φ) er sfæriske polarkoordinater på sfæren.

Solution : We calculate

$$\mathbf{V}(x, y, z) = \frac{d}{dt}|_{t=0}\varphi_t(x, y, z) = (-z, 0, x) = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

We see that φ_t is a rotation of $\mathbb{R}^3$ around the y-axis (which is fixed), so it is an orthogonal transformation and hence preserves the length of vectors. In particular, the 2-sphere is invariant under the flow (it is rotated).

The last question is a bit tricky, since the length of the calculations depends very much on where we chooce the center of the spherical polar coordinates. Unfortunately, $\mathbf{V}$ generates rotations in the xz-plane, so it would be laborious to use the usual coordinates centered at a pole (0, 0,1). However, if we choose the center at the pole (0,1, 0) on the y-axis, say

$$z = \sin\varphi\cos\theta,\ x = \sin\varphi\sin\theta, y = \cos\varphi,$$

then we can show that $\mathbf{V}|_{S^2} = \pm\frac{\partial}{\partial\theta}$. Geometrically, it is clear that the flow in terms of (θ, φ) is just translation in the direction of θ and with φ kept constant, namely

$$\varphi_t(\theta, \varphi) = (\theta \pm t, \varphi)$$

where the choice of sign of $\pm t$ depends on choice of positive direction of θ.