Riemannian geometry — a brief introduction
`(for TMA 4190 Manifolds, spring 2011)`

In differential topology one associates to a manifold M its tangent bundle TM, cotangent bundle T^*M (as well as other tensor bundles), and various other invariants, but these invariants cannot be used to distinguish manifolds which are diffeomorphic. However, in Riemannian geometry another kind of structure is imposed on the manifold, which is of a geometric nature, namely a *Riemannian metric*. The new structure enables one to talk about distances, angles, length, area, volume, etc., The principal example is, indeed, an old friend, namely *Euclidean geometry* . Thus, in the Euclidean plane E^2 we can distinguish between a circle and an ellipse, for example, and also between circles of different radius. But we cannot see any difference between isometric "figures" or isometric manifolds. Riemannian geometry is the theory which enables us to study manifolds from a geometric viewpoint, in a way which is as close to Euclidean geometry as possible.

In fact, in a very small neighborhood of any point the geometry is approximately Euclidean. We need to be more precise about this. First of all, by an inner product on a vector space V we mean a positive definite symmetric bilinear form $\langle \mathbf{u}, \mathbf{v} \rangle$. This enables one to introduce geometric concepts in V such as the length of a vector, distance between points, angle between rays, area and so on, in the space V. Here is a precise definition of how to generalize all this to manifolds.

Definition 1 *A Riemannian metric on a smooth manifold is an assignment of an inner product $\langle \mathbf{u}, \mathbf{v} \rangle_p$ in each tangent space T_pM. Moreover, for any two smooth vector fields X, Y on M, the function $M \to \mathbb{R}$ defined by $p \to \langle X_p, Y_p \rangle_p$ is also smooth.*

The last condition is needed for the metric to be smooth, loosely speeking ,so that the inner product changes smoothly from point to point. We shall say more about this later. Before we continue, we shall have a closer look at the simplest type of smooth manifolds, namely the Cartesian n-space $\mathbb{R}^n$ which serves as the local model for any other n-dimensional manifold.

0.1 Cartesian n-space, global and local coordinates

The *Cartesian n-space* $\mathbb{R}^n = \{(x_1, x_2, ..., x_n); x_i \in \mathbb{R}\}$ consists of all n-tuples of real numbers. It is naturally an n-dimensional vector space over $\mathbb{R}$, with the zero vector as a distinguished point O called the origin. There is also an obvious basis

$$\mathbf{e}_1 = (1, 0, 0, .., 0), \mathbf{e}_2 = (0, 1, 0, .., 0), ..., \mathbf{e}_n = (0, ..., 0, 1),$$

often referred to as the *standard basis.* $\mathbb{R}^n$ serves as the standard model for all n-dimensional real vector spaces, since they are all isomorphic to $\mathbb{R}^n$ and the latter is evidently the simplest example. More precisely, any n-dimensional vector space V with a given basis $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$ can be immediately identified with $\mathbb{R}^n$ via the following isomorphism

$$\sum_i x_i \mathbf{b}_i \to (x_1, x_2, ..., x_n)$$

In differential topology we study manifolds of a given dimension n, and they are locally modelled on $\mathbb{R}^n$ (or any n-dimensional real vector space). In particular, $\mathbb{R}^n$ is the simplest example of an n-dimensional manifold, with the differentiable structure determined by the atlas consisting of a single chart, namely the identity map on $\mathbb{R}^n$.

A special feature of a manifold diffeomorphic to $\mathbb{R}^n$ is that we may simply replace it by $\mathbb{R}^n$ and thus it also becomes a vector space. Assuming this has been done, points P are identified with their position vector $\overrightarrow{OP}$, on which we can perform vector algebra and calculus calculations – well known from linear algebra and analytic geometry in n-space.

For a smooth manifold M in general, the tangent spaces T_pM are vector spaces, and they are isomorphic because they have the same dimension. But in general there is no natural isomorphism between the tangent spaces at different points. However, when $M = \mathbb{R}^n$ each vector space $T_p\mathbb{R}^n$ is naturally isomorphic to $\mathbb{R}^n$ itself, essentially because position vectors in $\mathbb{R}^n$ and tangent vectors can be viewed as vectors in the same vector space $\mathbb{R}^n = M$. This is a very special property which only makes sense for the manifolds $\mathbb{R}^n$ (or open subsets of them). As a consequence, given any basis $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$ of $\mathbb{R}^n = M$, these vectors also represent global vector fields

$$\mathbf{b}_i : \mathbb{R}^n \to T\mathbb{R}^n, \;\; p \to \mathbf{b}_i|_p \in T_p\mathbb{R}^n \simeq \mathbb{R}^n$$

To such a basis corresponds a system of *Cartesian coordinates* $(y_1, y_2, ..., y_n)$ on M, as follows

$$p = \sum y_i \mathbf{b}_i \longleftrightarrow (y_1, y_2, ..., y_n)$$

In differential topology these vector fields are often denoted

$$\mathbf{b}_1 = \frac{\partial}{\partial y_1}, ..., \mathbf{b}_n = \frac{\partial}{\partial y_n} \tag{1a}$$

and we call them the *coordinate vector fields* associated with the coordinates $y_1, y_2, ...y_n$.Clearly, any smooth vector field on M can be expressed as a unique linear combination

$$X = f_1 \frac{\partial}{\partial y_1} + f_2 \frac{\partial}{\partial y_2} + ... + f_n \frac{\partial}{\partial y_n}$$

where the f_i are smooth functions.

Here we point out that Cartesian coordinates on $\mathbb{R}^n$ is just a very special type of coordinates, closely related to the vector space structure . But sometimes one also need to work with more general coordinate systems on $\mathbb{R}^n$, perhaps only defined on some open subset U. Given n functions u_i on U, they yield a coordinate system $(u_1, ..., u_n)$ iff the map

$$p \to \phi(p) = (u_1(p), ..., u_n(p)) \tag{2}$$

is a diffeomorphism between U and some open subset $\phi(U)$ of $\mathbb{R}^n$. To this coordinate system there is associated vector fields on U

$$\frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_n} \tag{3}$$

which we also refer to as the coordinate vector fields. Thus they provide a basis $\frac{\partial}{\partial u_1}|_p, ..., \frac{\partial}{\partial u_n}|_p$ for the tangent space T_pM , for each $p \in U$. To make sense out of the symbol $\frac{\partial}{\partial u_i}$, recall the usual definition of a tangent vector in a manifold. Namely, a vector field (resp. tangent vector) can be interpreted as a derivation (resp. local derivation) acting on the ring of smooth functions. So it should be clear what we mean by the operator $\frac{\partial}{\partial u_i}$.

Still, one may find all this somewhat confusing, whether the given manifold M is $\mathbb{R}^n$ or not. Namely, suppose we have a chart (ϕ, U) on a given manifold M, say the function

$$\phi : U \to \mathbb{R}^n$$

is given by (2), where $(u_1, u_2, ..., u_n)$ are Cartesian coordinates on the (target) space $\mathbb{R}^n$. Then the latter space has coordinate vector fields $\frac{\partial}{\partial u_i}$ as explained before, but on the other hand we also use the same notation for the "induced" vector fields on $U \subset M$. The reason for this is that we want the notation as simple as possible. But in fact, there is a linear isomorphism $\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n$ so that $\frac{\partial}{\partial u_i}|_p$ denotes the vector which is mapped to $\frac{\partial}{\partial u_i}|_{\phi(p)}$. The latter acts on functions F on $\mathbb{R}^n$ (or a nbd of $\phi(p)$) as follows

$$F \to \frac{\partial F}{\partial u_i}(\phi(p) \text{ , partial differentiation with respect to } u_i,$$

whereas in the manifold M the tangent vector $\frac{\partial}{\partial u_i}|_p$ acts on functions f (defined on a nbd of p) by

$$f \to \frac{\partial(f \circ \phi^{-1})}{\partial u_i}(\phi(p), \text{ also denoted } \frac{\partial f}{\partial u_i}(p) \text{ for brevity}$$

Example 2 *Consider $M = \mathbb{R}^3$ with the Cartesian coordinates (x, y, z) on the one hand, and spherical coordinates (ρ, φ, θ) on the other hand, defined by the relations*

$$x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi \tag{4}$$

The spherical coordinates are not globally defined, so let's assume working on some open set U where the equations (4) can be solved for ρ, φ, θ in terms of x, y, z.From the equations (4) one can immediately calculate the coordinate covector fields dx, dy, dz on U expressed as linear combinations of the coordinate *covector fields $d\rho, d\varphi, d\theta$, namely by differentiation*

$$\begin{aligned} dx &= \sin\varphi\cos\theta d\rho + \rho\cos\varphi\cos\theta d\varphi - \rho\sin\varphi\sin\theta d\theta \\ dy &= \sin\varphi\sin\theta d\rho + \rho\cos\varphi\sin\theta d\varphi + \rho\sin\varphi\cos\theta d\theta \\ dz &= \cos\varphi d\rho - \rho\sin\varphi d\varphi \end{aligned} \tag{5}$$

In order to express $d\rho, d\varphi, d\theta$ in terms of dx, dy, dz, you must either invert the relations (4) and differentiate as above, or you can (much easier?) solve the linear system (5).

Problem 3 *In the previous example, express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ as a linear combination of $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \theta}$, and conversely.*

What you should know is that the covector basis dx, dy, dz (at a given point p) is dual to the vector basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$(at p), and similarly with the other coordinate system. The system (5) can be formally written in terms of matrices as

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = A \begin{pmatrix} d\rho \\ d\varphi \\ d\theta \end{pmatrix} \tag{6}$$

where A is the Jacobi matrix of the transformation $(\rho, \varphi, \theta) \to (x, y, z)$, having determinant $\rho^2 \sin\varphi$, namely

$$\begin{aligned} A &= \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} \\ A^{-1} &= \begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ \frac{1}{\rho}\cos\theta\cos\varphi & \frac{1}{\rho}\cos\varphi\sin\theta & -\frac{1}{\rho}\sin\varphi \\ -\frac{1}{\rho}\frac{\sin\theta}{\sin\varphi} & \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi} & 0 \end{pmatrix} \end{aligned}$$

In general, it follows from simple linear algebra that if a relation like (6) holds for two bases in a 3-dim vector space V, then there is a corresponding relation between their dual bases in V^*, but in the "opposite" direction and with the transpose matrix, namely we have

$$\begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = A^T \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (A^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

In particular, we write out for later reference

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\sin\varphi\frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos\theta\cos\varphi\frac{\partial}{\partial \varphi} - -\frac{1}{\rho}\frac{\sin\theta}{\sin\varphi}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\sin\varphi\frac{\partial}{\partial \rho} + \frac{1}{\rho}\cos\varphi\sin\theta\frac{\partial}{\partial \varphi} + \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} &= \cos\varphi\frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial \varphi} \end{aligned} \tag{7}$$

0.2 The Euclidean n-space

As a smooth manifold, the Euclidean n-space E^n is just an n-dimensional vector space, so let us say $\mathbb{R}^n$. One can describe the geometry by a system of axioms (concerning points, lines, planes, angles, parallelism, etc.), and this was also the original approach dating back to ancient times more than 2000 years ago. But nowadays we can also say that E^n is a smooth manifold with a "flat" Riemannian structure. In fact, E^n is the simplest n-dimensional Riemannian manifold one can imagine.

Vector algebra (or vector geometry) is the algebraization of Euclidean geometry, which makes it possible to calculate and prove geometric theorems by algebraic and analytic methods. The identification of E^n with $\mathbb{R}^n$ is the coordinate approach to geometry, which dates back to Descartes and Fermat in the 17th century, from which analytic geometry arose. So it became customary to identify E^n with the Cartesian n-space $\mathbb{R}^n$, namely an n-dimensional vector space equipped with some geometric structure. From a modern viewpoint, all we need extra is an *inner product* $\langle \mathbf{u}, \mathbf{v} \rangle$, from which one can define the length of line segments and angles between crossing lines. In some sense, the inner product is the algebraization of *Pythagoras theorem* in Euclidean geometry, which tells us that in a triangle with a right angle, the three sides a, b, c satisfy the identity $a^2 + b^2 = c^2$ where c is the length of the side opposite to the right angle. This property is, in fact, just a reformulation of the *Euclidean parallel postulate* which says that the Euclidean plane has the following property: given a line L and a point P outside L , then there is a unique line L' parallel to L and passing through P. The property is also equivalent to saying that the sum of the three angles in a triangle is equal to two right angles (i.e., 180 degrees). It is this property that fails in hyperbolic geometry, as well as in spherical (also called elliptic) geometry.

Now, having an inner product in $E^n = \mathbb{R}^n$, we have in fact the same inner product $\langle \mathbf{u}, \mathbf{v} \rangle_p$ in each tangent space $T_p\mathbb{R}^n \simeq \mathbb{R}^n$. Therefore, if $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ are the coordinate vector fields corresponding to Cartesian coordinates $(x_1, ..., x_n)$, then the functions

$$g_{ij}(p) = \left\langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \right\rangle_p \tag{8}$$

are by definition constant, that is, independent of p.This is a characteristic property of Euclidean geometry.

In a general Riemannian manifold one may not be able to find any coordinate system, in a nbd of a given point p, such that the functions g_{ij} are constant. That happens when the curvature (or curvature tensor) is non-vanishing at p. But otherwise, if they are constant then we say that the space is flat near p. But besides E^n itself and open subsets of it, are there other flat n-dim Riemannian manifolds? Well, the space must be locally isometric to E^n, but how can it look like globally? It is certainly difficult to imagine a compact flat surface imbedded in E^3.

We shall not give the definition of curvature, but any compact surface in 3-space must obviously have positive curvature somewhere in order to close up to a closed and bounded set. However, Riemannian manifolds have their intrinsic geometry, and it is for example possible to give the 2-dimensional torus $S^1 \times S^1$ a Riemannian flat metric. The discovery of the "flat torus" was a big surprise around 1870, but in modern mathematics the construction is quite easy. Namely, a 2-dim torus is a quotient space of $\mathbb{R}^2$ (show this!) and the quotient map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ is a local diffeomorhism. This allows us to "push" down the metric on $\mathbb{R}^2$ to a metric on the torus, which makes the quotient map a local isometry . But being flat is a local property, so being flat in a nbd of every point, the torus is certainly flat overywhere.

There is the following theorem which characterize the Riemannian manifold E^n :

Theorem 4 *The Riemannian manifold M is isometric to the Euclidean n-space E^n if and only if it is diffeomorphic to $\mathbb{R}^n$ and, moreover, there are n vector fields $X_1, X_2, ..., X_n$ which are linear independent at each point and such that the functions $p \to \langle X_i(p), X_j(p) \rangle_p$ are constant for all i, j.*

0.3 Riemannian manifolds

A Riemannian manifold M has an inner product $\langle \mathbf{u}, \mathbf{v} \rangle_p$ on each tangent space. Therefore, given local coordinates $(x_1, ..., x_n)$ in a nbd U the metric is uniquely determined in U by the following functions on U

$$p \to g_{ij}(p) = \left\langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \right\rangle_p$$

Thus the matrix function (g_{ij}) determines the metric in the nbd U. By introducing the arc-length element ds the metric on U is also classically written as

$$ds^2 = \sum g_{ij} dx_i dx_j \tag{9}$$

The reason for this can be explained by considering a parametrized smooth curve $t \to \gamma(t)$. Its velocity field along the curve is

$$\frac{d\gamma}{dt} = \frac{dx_1}{dt} \frac{\partial}{\partial x_i} + ... + \frac{dx_n}{dt} \frac{\partial}{\partial x_n}$$

and the speed $v(t) = \frac{ds}{dt}$ is the length of the velocity vector, namely

$$v^2 = ||\frac{d\gamma}{dt}||^2 = \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = \sum g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \tag{10}$$

Multiplying v^2 by dt^2 we get ds^2 , so equation (9) is really the identity (10) without any specific parametrization of the curve.

The length of the curve from $t = a$ to $t = b$ is

$$L(\gamma) = \int ds = \int_a^b v dt = \int_a^b \sqrt{\sum g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt}} dt$$

In fact, the length does not depend on the choice of parametrization. Now we can also introduce a distance function by saying that the distance $\delta(p,q)$ from p to q is the length of the shortest piecewice smooth curve from p to q. Well, here is a problem, since there may not be any curve with the shortest length. So we rather define

$$\delta(p,q) = \inf\{L(\gamma); \gamma \text{ piecewise smooth and connects } p \text{ and } q\}$$

This makes M to a metric space, and fortunately the topology defined by the distance function δ coincides with the original topology on the manifold!

We also mention that so-called *geodesic curves* are the generalization of straight lines in Euclidean geometry. For two nearby points on such a curve, the curve segment between the points is the unique shortest curve and hence its length gives the distance. However, globally, this may not hold. That is, when the points are moved further away from each other, the geodesic segment between them may cease to be the shortest curve between the points. A simple example is the round 2-sphere S^2, where the geodesics are the great circles. Take a look and see what happens.

Definition 5 *Two Riemannian manifolds M and N are isometric if there is a diffeomorphism $\phi : M \to N$ such that the induced linear isomorphism $\phi_* : T_pM \to T_{\phi(p)}N$ is an isometry (i.e. preserves the inner product of vectors) for each point p.*

The Riemannian metric on a manifold M enables us to construct an isomorphism between the tangent bundle and the cotangent bundle

$$TM \to \begin{array}{ccc} TM & \simeq & T^*M \\ \downarrow & & \downarrow \\ M & = & M \end{array} \tag{11}$$

This follows from a well known result in linear algebra, namely an inner product on a vector space V enables one to identify an element α of V^*, that is, a linear functional $\alpha : V \to V^*$, with a vector $\bar{\alpha} \in V$, as follows

$$\alpha(\mathbf{v}) = \langle \bar{\alpha}, \mathbf{v} \rangle$$

Conversely, any vector $\mathbf{u}$ in V corresponds to an element $\bar{\mathbf{u}}$ in V^*, namely the functional $\mathbf{v} \to \langle \mathbf{u}, \mathbf{v} \rangle$. In physics one likes to write $\langle \alpha, \mathbf{v} \rangle = \alpha(\mathbf{v})$ since after all, applying α to $\mathbf{v}$ gives a bilinear pairing $V^* \times V \to \mathbb{R}$, even when there is no inner product involved. Thus, we can write $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \bar{\mathbf{u}}, \mathbf{v} \rangle = \bar{\mathbf{u}}(\mathbf{v})$ etc.

Recall that any smooth function f on M gives a covector field df, called the differential of f. A covector field is also called a 1-form. Now, with reference to the diagram (11), let us replace the projection "arrow" on the right by an upward going "arrow" representing df . Then, composing the function df with the diffeomorphism $T^*M \to TM$ we obtain a cross section $M \to TM$, that is, a vector field which we shall denote by ∇f . This is the *gradient field* of f. In other words, the gradient of f at p is the vector $\nabla f(p)$ in T_pM defined by the property that

$$\langle \nabla f(p), \mathbf{v} \rangle_p = \langle df|_p, \mathbf{v} \rangle = df(\mathbf{v}) = \mathbf{v}(f) \tag{12}$$

holds for every vector $\mathbf{v}$ in T_pM.

Problem 6 *Find the expression for the gradient ∇F of a function on Euclidean 3-space in terms of spherical coordinates. Let f be a function on the round 2-sphere S^2 of radius 1. Calculate the gradient of f using spherical coordinates.*

Let us see what to do. In E^3 the Euclidean coordinates (x, y, z) has coordinate fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ which constitute an orthonormal basis at every point, and hence

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{13}$$

Everybody is told that the gradient of F is given by

$$\nabla F = \frac{\partial F}{\partial x}\frac{\partial}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial}{\partial z} \tag{14}$$

and this is correct. Therefore,since the vectors $\frac{\partial}{\partial x}$, .. have already been expressed in spherical coordinates in (7), we need only substitute into (14).

On the other hand, we obtain the metric (13) in spherical coordinates by substituting the expressions (5) into the expression (13) for ds^2, which yields

$$ds^2 = d\rho^2 + \rho^2(d\varphi^2 + \sin^2\varphi d\theta^2) \tag{15}$$

Then we can calculate directly the gradient of F from the metric (15) if we use (12). To illustrate this , let us write

$$\nabla F = a\frac{\partial}{\partial \rho} + b\frac{\partial}{\partial \varphi} + c\frac{\partial}{\partial \theta}$$

and try to determine a, b, c. Let $\mathbf{v} = \alpha\frac{\partial}{\partial \rho} + \beta\frac{\partial}{\partial \varphi} + \gamma\frac{\partial}{\partial \theta}$ be any vector, and calculate the inner product

$$\begin{aligned}\langle \nabla F, \mathbf{v} \rangle &= a\alpha + b\beta\rho^2 + c\gamma\rho^2\sin^2\varphi = \mathbf{v}(F) \\ &= \frac{\partial F}{\partial \rho}\alpha + \frac{\partial F}{\partial \varphi}\beta + \frac{\partial F}{\partial \theta}\gamma\end{aligned}$$

Since this must hold for all numbers α, β, γ, we conclude that

$$a = \frac{\partial F}{\partial \rho}, \quad b = \frac{1}{\rho^2}\frac{\partial F}{\partial \varphi}, \quad c = \frac{1}{\rho^2\sin^2\varphi}\ \frac{\partial F}{\partial \theta}$$

Finally, consider the function f on the 2-sphere $S^2 = (\rho = 1)$ in E^3. In the metric expression (15), put $\rho = 1$ and obtain the spherical metric on S^2, as a sub-Riemannian manifold. Its metric in spherical coordinates (φ, θ) is

$$ds^2 = d\varphi^2 + \sin^2\varphi d\theta^2 \tag{16}$$

and calculation of the gradient yields the result

$$\nabla f = \frac{\partial f}{\partial \varphi}\frac{\partial}{\partial \varphi} + \frac{1}{\sin^2 \varphi}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}$$

Let us return to the identites (7). In particular , the unit vector in the positive z-axis direction, at points outside the z-axis, decomposes as

$$\mathbf{k} = \frac{\partial}{\partial z} = \cos\varphi \frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\varphi \frac{\partial}{\partial \varphi}$$

Let p be a point on the unit sphere S^2, so we set $\rho = 1$.The vector $\frac{\partial}{\partial \rho}$ at p is a unit vector in 3-space which is normal to the sphere, so along S^2 the field $\frac{\partial}{\partial \rho}$ is a normal vector field pointing outward. However, the vectors $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial \theta}$ are perpendicular to $\frac{\partial}{\partial \rho}$, so they are tangential to the sphere and therefore they constitute a basis for T_pS^2 at the point p. The basis is orthogonal but not orthonormal. In fact, from (16) it follows that $\frac{\partial}{\partial \varphi}$ has length 1 and $\frac{\partial}{\partial \theta}$ has length $\sin\varphi$. This also explains why spherical coordinates are singular along the z-axis. On S^2 we see that the coordinate vector $\frac{\partial}{\partial \theta}$ becomes zero at the north and south pole, but away from the poles S^2 has the following two orthonormal vector fields

$$\{\frac{\partial}{\partial \varphi},\ \frac{1}{\sin\varphi}\frac{\partial}{\partial \theta}\}$$