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Eksam in TMA4190 Manifolds

English
Thursday June 9, 2011
Time : 09.00–13.00

SOLUTIONS

Permitted aids : Code D
Grades: June 30 , 2011

Information In the problems below, the term differentiable (or smooth) map or manifold means C^∞-differentiable. There are altogether 10 subproblems, all of equal importance.

Problem 1

a) Let S be the figure ∞ (figure eight) in the plane $\mathbb{R}^2$. Answer the following questions, with a brief justification.

(i) : Is S a compact subset of the plane?

(ii) : Is S the image of $\mathbb{R}$ by an immersion ?

(iii) : Is S a 1-dimensional submanifold of the plane ?

b) Let $M \subset \mathbb{R}^4$ be defined by the equations

$$2x + 2y + az = 0,\ 2xy + 3w = 0$$

where a is a constant. Show that M is a 2-dimensional smooth manifold.

Solution

a) (i) S is a closed and bounded subset of the Euclidean plane, and these are precisely the compact subsets (cf. Heine-Borel).

(ii) Clearly, it is possible to find a smooth parametrization $t \to \gamma(t)$ of the "figure eight" such that the velocity vector $\gamma'(t)$ is never zero, and then the map $t \to \gamma(t)$ will be an immersion of $\mathbb{R}$ (or if you like, of an interval or a circle) into the plane.

(iii) S is not locally Euclidean everywhere. Namely, at the middle point the curve intersects itself, so no nbd of this point can be homeomorphic to an open interval (or open set of $\mathbb{R}$). In particular, S cannot be a submanifold of the plane.

b) We consider the map $F : \mathbb{R}^4 \to \mathbb{R}^2$

$$F : (x, y, z, w) \to (2x + 2y + az, 2xy + 3w)$$

The Jacobi matrix is

$$D(F) = \begin{pmatrix} 2 & 2 & a & 0 \\ 2y & 2x & 0 & 3 \end{pmatrix}$$

It is easy to verify that the matrix has (maximal) rank 2 at each point in $\mathbb{R}^4$. For example, the first and last column make a 2×2 -submatrix of determinant 6, irrespective of the value of a. So the map is a submersion, and therefore all values (i.e. points $q \in \mathbb{R}^2$) are regular. It follows that $F^{-1}(q)$ is a smooth submanifold of $\mathbb{R}^4$ of dimension $4 - 2 = 2$, for any q. In particular, $M = F^{-1}(0, 0)$ is a 2-dimensional smooth manifold.

Problem 2

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping

$$(x, y, z) \to (x', y', z') = (e^{2y} + e^{2z}, e^{2x} - e^{2z}, x - y)$$

a) Calculate the Jacobi-matrix of φ. Use this to explain why the image set $\varphi(\mathbb{R}^3)$ is open and, furthermore, that φ is a local diffeomorphism.

b) Show that $\varphi : \mathbb{R}^3 \to \varphi(\mathbb{R}^3)$ is a diffeomorphism.

Solution

(a) The Jacobi matrix is

$$D(\varphi) = \begin{pmatrix} 0 & 2e^{2y} & 2e^{2z} \\ 2e^{2x} & 0 & -2e^{2z} \\ 1 & -1 & 0 \end{pmatrix}$$

Its determinant is $4(e^{2x}e^{2z} + e^{2y}e^{2z}) > 0$, so the matrix is invertible at each point $p \in \mathbb{R}^3$. By the Inverse Function Theorem , φ is invertible in a nbd of any point p, and moreover, a small open nbd of p is mapped to an open nbd of $\varphi(p)$. Namely, φ is a local diffeomorphism (everywhere). In particular, the image set $\varphi(\mathbb{R}^3)$ is a union of open sets and is therefore an open set.

(b) Still, we dont know whether φ is (globally) injective. So, let us take a closer look at the system of equations

$$e^{2y} + e^{2z} = x', \; e^{2x} - e^{2z} = y', \; x - y = z' \tag{1}$$

We claim that for given values of x', y', z', there is (at most) one solution (x, y, z), and this will also prove that φ is a diffeomorphism onto its image. First of all, $x = y + z'$, so y will determine x. Now, by adding the first two equations of (1) we obtain

$$\begin{aligned} e^{2y} + e^{2x} &= x' + y' \\ e^{2y}(1 + e^{2z'}) &= x' + y' \\ y &= \frac{1}{2} \ln(\frac{x' + y'}{(1 + e^{2z'})}) \end{aligned}$$

Finally, we can determine z from the first or second equation of (1), in terms of (x', y', z').

Problem 3

Let $P(x_1, x_2, .., x_k)$ be a polynomial of k variables, which is homogeneous and of degree $m > 0$, namely

$$P(tx_1, tx_2, .., tx_k) = t^m P(x_1, x_2, .., x_k)$$

We shall make use of the following property of homogeneous functions expressed by Euler's identity:

$$\sum_{i=1}^{k} x_i \frac{\partial P}{\partial x_i} = mP \qquad (2)$$

a) Show that 0 is the only critical value of P.

b) Show that for $a \neq 0$ is

$$P^{-1}(a) = \left\{x \in \mathbb{R}^k; P(x) = a\right\}$$

a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$. Show also that $P^{-1}(a)$ and $P^{-1}(b)$ are diffeomorphic when $ab > 0$. (Hint : think geometrically, and find a suitable geometric transformation $\mathbb{R}^k \to \mathbb{R}^k$).

Solution

a) Regard P as a map $\mathbb{R}^k \to \mathbb{R}$, with Jacobi matrix (or gradient vector ∇P)

$$D(P) = (\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, ..., \frac{\partial P}{\partial x_k})$$

By definition, a critical point $p \in \mathbb{R}^k$ is a point where $D(P)$ vanishes, namely the rank of $D(P)$ is zero, and then $P(p)$ is a critical value.The left side of (2) is, in fact, the inner product of ∇P with the position vector $p = (x_1, ..., x_k)$. Hence, if ∇P is zero at p, then the inner product vanishes, so clearly $P(p) = 0$ on the right side of (2). Thus, 0 is the only critical value.

b) We have seen that any $a \neq 0$ is a regular value for P, consequently $P^{-1}(a)$ is a smooth submanifold of $\mathbb{R}^k$ of dimension $k - 1$.

Note the simplest case $m = 1$, when $P^{-1}(a)$ is just a hyperplane in k-space $\mathbb{R}^k$, and the case $m = 2$, when $P^{-1}(a)$ is a quadratic hypersurface (also called a quadric), namely higher dimensional analogs of conic sections. Imagine for example, the case $m = 2, k = 3$, and $P^{-1}(a)$ an ellipsoid. Then for $c > 0$, $P^{-1}(ca)$ is an ellipsoid of the same shape, but of different size.

Now, for $t > 0$, consider the following map

$$\mu_t : \mathbb{R}^k \to \mathbb{R}^k, \ (x_{1,}..., x_k) \to (tx_1, ..., tx_k)$$

Then we see that

$$P(x_1, ..., x_k) = a \Longleftrightarrow P(tx_1, ..., tx_k) = \ at^m$$

Setting $at^m = b$, it follows that μ_t maps $P^{-1}(a)$ bijectively to $P^{-1}(b)$. Conversely, if a, b are given and $ab > 0$, we set $t = \sqrt[m]{b/a}$ and define the map μ_t as before. Finally, observe that μ_t is linear and invertible, and is therefore a diffeomorphism. In Euclidean geometry, μ_t is also called a *homothety.* It preserves the "shape" of a geometric figure, but not its size (unless $t = 1$).

Problem 4

Let $\mathbb{R}^3$ be the Euclidean 3-space with cartesian coordinates (x, y, z) and Riemannian metric on standard form $ds^2 = dx^2 + dy^2 + dz^2$, and let Σ denote the sylinder given by the equation $x^2 + y^2 = 1$ and with the induced metric $d\sigma^2 = ds^2|_\Sigma$.

a) Show that the surface $(\Sigma, d\sigma^2)$ is locally flat, that is, each point has a neighborhood U with local coordinates (u, v) such that $d\sigma^2 = du^2 + dv^2$ on U.

Note U can therefore be identified with an open subset of the Euclidean uv-plane, which may simplify the calculation of the distance between given points in U.

b) The points $p_1 = (1, 0, 0)$ and $p_2 = (0, 1, 2)$ lie on Σ. Determine the distance between p_1 and p_2 relative to Σ, that is, the length of the shortest curve on Σ between the two points.

Solution

a) The sylinder is circular with the z-axis as symmetry axis. It cuts the xy-plane in the circle $x^2 + y^2 = 1$. It is natural to introduce sylinder coordinates (r, θ, z), defined by

$$x = r\cos\theta, \;\; y = r\sin\theta$$

Then the sylinder Σ is simply given as the "coordinate surface" $r = 1$. Let us calculate the metric expression ds^2 restricted to the sylinder Σ, using the fact that $r = 1$ and $dr = 0$ on Σ :

$$\begin{aligned} d\sigma^2 &= ds^2|_\Sigma = (dr\cos\theta - r\sin\theta d\theta)^2 + (dr\sin\theta + r\cos\theta d\theta)^2 + dz^2 \\ &= d\theta^2(\sin^2\theta + \cos^2\theta) + dz^2 = d\theta^2 + dz^2 \end{aligned}$$

So, we put $u = \theta$ and $v = z$ and obtain a coordinate system (u, v) on the sylinder with the appropriate properties.

b) For u and v suitably limited so that (u, v) becomes a coordinate system on a region U of Σ, the metric expression $d\sigma^2 = du^2 + dv^2$ on U tells us that the geometry on U is just the geometry of the corresponding region U' of the Euclidean uv-plane. In particular, the distance between points $q_1 = (u_1, v_2)$ and $q_2 = (u_2, v_2)$ in U' is given by the usual formula

$$d(q_1, q_2) = \sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2}$$

Hence, if p_1 and p_2 are the corresponding points in U, their distance in Σ will also be $d(q_1, q_2)$.

The given points $p_1 = (1, 0, 0)$ and $p_2 = (0, 1, 2)$ belong to such a (u, v)-coordinate neighborhood U. The (θ, z)-coordinates of p_1 and p_2 are $(0, 0)$

and $(\pi/2, 2)$, respectively. So we have $(u_1, v_1) = (0, 0), (u_2, v_2) = (\pi/2, 2)$, consequently the distance between the two points on Σ is

$$d(p_1, p_2) = \sqrt{(\pi/2 - 0)^2 + (2 - 0)^2} = \sqrt{\frac{\pi^2}{4} + 4}$$

Problem 5

We introduce two 1-parameter groups of transformations of the xy-plane $\mathbb{R}^2$:

$$\varphi_t : (x, y) \rightarrow (x + t, y),\ \psi_t : (x, y) \ \rightarrow ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y)$$

where φ_t is generated by the vector field X and ψ_t is generated by the vector field Y.

a) Determine the vector fields X, Y and the commutator product (or Lie-product) $[X, Y]$, all expressed on the form $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$.

b) Let $p = (x, y)$ be an arbitrary point and consider the curve γ through $p = \gamma(0)$ parametrized by

$$t \rightarrow \gamma(t) = \psi_{-t} \circ \varphi_{-t} \circ \psi_t \circ \varphi_t(p)$$

Calculate $\gamma(t)$ and verify the following:

$$\frac{d\gamma}{dt}(0) = \mathbf{0},\ \frac{d^2\gamma}{dt^2}(0) = 2[X, Y]_p$$

Solution

a) Write $p = (x, y)$. Then

$$X_p = \frac{d}{dt}|_{t=0}(\varphi_t(p)) = \frac{d}{dt}|_{t=0}(x + t, y) = (1, 0)_p \longleftrightarrow \frac{\partial}{\partial x}|_p$$
$$Y_p = \frac{d}{dt}|_{t=0}(\psi_t(p)) = (-y, x)_p \longleftrightarrow -y\frac{\partial}{\partial x}|_p + x\frac{\partial}{\partial y}|_p$$

So we have the vector fields

$$X = \frac{\partial}{\partial x},\quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

To calculate $[X, Y]$, let us regard this as an operator on functions f, and calculate

$$\begin{aligned}[X, Y]f &= XYf - YXf = X(Yf) - Y(Xf)\\ &= X(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}) - Y(\frac{\partial f}{\partial x})\\ &= \frac{\partial}{\partial x}(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}) - (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\frac{\partial f}{\partial x}\\ &= \frac{\partial f}{\partial y} + (x\frac{\partial^2 f}{\partial x\partial y} - y\frac{\partial^2 f}{\partial x^2}) - (x\frac{\partial^2 f}{\partial y\partial x} - y\frac{\partial^2 f}{\partial x^2})\\ &= \frac{\partial f}{\partial y}\end{aligned}$$

Since this holds for any f we conclude that $[X,Y] = \frac{\partial}{\partial y}$, which is the coordinate vector field in the direction of the y-axis.

b) We shall calculate $\gamma(t) = \psi_{-t} \circ \varphi_{-t} \circ \psi_t \circ \varphi_t(x,y)$, but let us represent points $p = (x,y)$ in $\mathbb{R}^2$ as column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\gamma(t) : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+t \\ y \end{pmatrix} \to \begin{pmatrix} (\cos t)(x+t) - (\sin t)y \\ (\sin t)(x+t) + (\cos t)y \end{pmatrix}$$
$$\to \begin{pmatrix} (\cos t)(x+t) - (\sin t)y - t \\ (\sin t)(x+t) + (\cos t)y \end{pmatrix}$$
$$\to \begin{pmatrix} (\cos t)[(\cos t)(x+t) - (\sin t)y - t] + (\sin t)[(\sin t)(x+t) + (\cos t)y] \\ -(\sin t)[(\cos t)(x+t) - (\sin t)y - t] + (\cos t)[(\sin t)(x+t) + (\cos t)y] \end{pmatrix}$$
$$= \begin{pmatrix} x + t - t\cos t \\ y + t\sin t \end{pmatrix}$$

$$\gamma'(t) = \begin{pmatrix} 1 - \cos t + t\sin t \\ \sin t + t\cos t \end{pmatrix}, \gamma'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$
$$\gamma''(t) = \begin{pmatrix} 2\sin t + t\cos t \\ 2\cos t - t\sin t \end{pmatrix}, \gamma''(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\frac{\partial}{\partial y}|_p = 2[X,Y]_p$$