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Eksam in TMA4190 Manifolds

English
Thursday June 9, 2011
Time : 09.00–13.00

Permitted aids : Code D
Grades: June xx , 2011

Information In the problems below, the term differentiable (or smooth) map or manifold means C^∞-differentiable. There are altogether 10 subproblems, all of equal importance.

Problem 1

a) Let S be the figure ∞ (figure eight) in the plane $\mathbb{R}^2$. Answer the following questions, with a brief justification.

(i) : Is S a compact subset of the plane?
(ii) : Is S the image of $\mathbb{R}$ by an immersion ?
(iii) : Is S a 1-dimensional submanifold of the plane ?

b) Let $M \subset \mathbb{R}^4$ be defined by the equations

$$2x + 2y + az = 0,\ 2xy + 3w = 0$$

where a is a constant. Show that M is a 2-dimensional smooth manifold.

Problem 2

Let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the mapping

$$(x, y, z) \rightarrow (x', y', z') = (e^{2y} + e^{2z}, e^{2x} - e^{2z}, x - y)$$

a) Calculate the Jacobi-matrix of φ. Use this to explain why the image set $\varphi(\mathbb{R}^3)$ is open and, furthermore, that φ is a local diffeomorphism.

b) Show that $\varphi : \mathbb{R}^3 \rightarrow \varphi(\mathbb{R}^3)$ is a diffeomorphism.

Problem 3

Let $P(x_1, x_2, .., x_k)$ be a polynomial of k variables, which is homogeneous and of degree $m > 0$, namely

$$P(tx_1, tx_2, .., tx_k) = t^m P(x_1, x_2, .., x_k)$$

We shall make use of the following property of homogeneous functions expressed by Euler's identity:

$$\sum_{i=1}^{k} x_i \frac{\partial P}{\partial x_i} = mP$$

a) Show that 0 is the only critical value of P.

b) Show that for $a \neq 0$ is

$$P^{-1}(a) = \{x \in \mathbb{R}^k; P(x) = a\}$$

a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$. Show also that $P^{-1}(a)$ and $P^{-1}(b)$ are diffeomorphic when $ab > 0$. (Hint : think geometrically, and find a suitable geometric transformation $\mathbb{R}^k \to \mathbb{R}^k$).

Problem 4

Let $\mathbb{R}^3$ be the Euclidean 3-space with cartesian coordinates (x, y, z) and Riemannian metric on standard form $ds^2 = dx^2 + dy^2 + dz^2$, and let Σ denote the cylinder given by the equation $x^2 + y^2 = 1$ and with the induced metric $d\sigma^2 = ds^2|_\Sigma$.

a) Show that the surface $(\Sigma, d\sigma^2)$ is locally flat, that is, each point has a neighborhood U with local coordinates (u, v) such that $d\sigma^2 = du^2 + dv^2$ on U.

Note U can therefore be identified with an open subset of the Euclidean uv-plane, which may simplify the calculation of the distance between given points in U.

b) The points $p_1 = (1, 0, 0)$ and $p_2 = (0, 1, 2)$ lie on Σ. Determine the distance between p_1 and p_2 relative to Σ, that is, the length of the shortest curve on Σ between the two points.

Problem 5

We introduce two 1-parameter groups of transformations of the xy-plane $\mathbb{R}^2$:

$$\varphi_t : (x, y) \to (x + t, y),\ \psi_t : (x, y) \ \to ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y)$$

where φ_t is generated by the vector field X and ψ_t is generated by the vector field Y.

a) Determine the vector fields X, Y and the commutator product (or Lie-product) $[X, Y]$, all expressed on the form $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$.

b) Let $p = (x, y)$ be an arbitrary point and consider the curve γ through $p = \gamma(0)$ parametrized by

$$t \to \gamma(t) = \psi_{-t} \circ \varphi_{-t} \circ \psi_t \circ \varphi_t(p)$$

Calculate $\gamma(t)$ and verify the following:

$$\frac{d\gamma}{dt}(0) = \mathbf{0}, \ \frac{d^2\gamma}{dt^2}(0) = 2[X, Y]_p$$