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Norwegian University of Science and Technology
Department of Mathematical Sciences

Exam in TMA4190 Manifolds

English
Wednesday June 2, 2010
Time : 09.00–13.00
SOLUTIONS

Permitted aids : Code D
Grades: June 23, 2010

Instructions. In the problems below, the term differentiable (or smooth) map or manifold means C^∞-differentiable.

Problem 1

a) Consider the cone

$$C = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = z^2\}$$

with the topology induced as a subset from $\mathbb{R}^3$. Is C a topological manifold ? Explain.

Solution C is a "double" cone in 3-space, with the common top point at origin. The surface C is the union of the two cones $z = \pm\sqrt{x^2 + y^2}$, and it is a connected topological space. However, if we remove the origin point, the two cones will be disconnected. That is, $C - \{0\}$ splits into the disjoint union of the upper surface with $z > 0$ and the lower surface with $z < 0$, so $C - \{0\}$ is disconnected (as a topological space with the subspace topology from $\mathbb{R}^3$.) On the other hand, a topological manifold is locally Euclidean, and clearly you cannot make a connected 2-dimensional manifold disconnected by just removing one point, not even locally. For example, if C was a manifold, there would be an open nbd in C around the origin which is homeomorphic to an open 2-disk, say. But removal of a point from this disk does not make it disconnected. Therefore, C is not a topological manifold.

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b) Show that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi : (x, y) \to (xe^y + y, xe^y - y)$$

is a diffeomorphism.

Solution The simplest way is to calculate explicitly the inverse of φ. Set

$$u = xe^y + y, \; v = xe^y - y$$

and solve for x and y in terms of u and v. We find first y and then x, namely

$$x = \frac{(u+v)}{2} e^{\frac{v-u}{2}} \;, \; y = \frac{(u-v)}{2}$$

Clearly, φ^{-1} is smooth, so φ is a diffeomorphism.

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Problem 2

a) Let M be a topological manifold. Explain briefly what is a smooth structure on M, and what is a smooth map $f : M \to M$.

Solution A smooth structure determines what continuous functions $f : M \to \mathbb{R}$ can be called "smooth". Formally, the smooth structure is uniquely determined by a maximal smooth atlas $\mathcal{A}$. An atlas $\mathcal{A}_0$ consists of charts $(\varphi_\alpha, U_\alpha)$, so that the union of the open sets U_α cover M and $\varphi_\alpha : U_a \to \bar{U}_\alpha$ is a homeomorphism onto an open subset of $\mathbb{R}^n$ (n is the dimension of M). $\mathcal{A}_0$ is said to be <u>smooth</u> if all the transision functions for overlapping charts,

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

(which are clearly homeomorphisms between open sets in $\mathbb{R}^n$) are diffeomorphisms. An atlas is maximal if whenever $(\varphi_\gamma, U_\gamma)$ is any chart on M which has smooth transitions (in the above sense) with all the charts in the atlas, the chart already belongs to the atlas. We note that a smooth atlas $\mathcal{A}_0$ on M is contained in a unique maximal smooth atlas $\mathcal{A}$, so that $\mathcal{A}_0$ determines a unique smooth structure on M.

The map $f : M \to M$ is said to be smooth if, loosely speaking, it is a smooth function when it is expressed in local coordinates. Namely, for any chosen atlas $\mathcal{A}_0 \subset \mathcal{A}$ and chosen two charts $(\varphi_\alpha, U_\alpha)$, (φ_β, U_β) belonging to $\mathcal{A}_0$ and such that $f^{-1}(U_\beta) \cap U_\alpha \neq \varnothing$, the composition

$$\bar{f}_{\alpha\beta} = \varphi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(f^{-1}(U_\beta) \cap U_\alpha) \to \varphi_\beta(U_\beta) = \bar{U}_\beta$$

is a smooth map (as a function between open sets in $\mathbb{R}^n$).

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b) Consider the homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x^5$. Let $\mathcal{A}$ denote the standard smooth structure of $\mathbb{R}$ and let $\mathcal{B}$ denote the smooth structure containing the chart $(\mathbb{R}, \varphi)$. Show that $\mathcal{A}$ is different from $\mathcal{B}$.

Solution $\mathcal{A}$ is defined by the atlas containing just the identity map $Id : \mathbb{R} \to \mathbb{R}$, as a globally defined chart. Similarly, $\varphi : \mathbb{R} \to \mathbb{R}$ is a chart belonging to $\mathcal{B}$. However, their transition function

$$Id \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}, \;\; x \to \sqrt[5]{x}$$

is not a diffeomorphism, since it is not even differentiable at $x = 0$.

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c) Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Describe the property f must have in order to be a smooth map $f : \mathbb{R} \to \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}$ denotes the real line $\mathbb{R}$ with the smooth structure $\mathcal{B}$. Find a map f which is a diffeomorphism between $\mathbb{R}$ and $\tilde{\mathbb{R}}$?

Solution : As above, for $\mathbb{R}$ we use the atlas consisting of the chart $(Id, \mathbb{R})$ and for $\tilde{\mathbb{R}}$ we use the atlas consisting of the chart $\varphi : \mathbb{R} \to \mathbb{R}$. Thus the local expression for f will be

$$\bar{f} = \varphi \circ f \circ Id^{-1} : x \to f(x)^5$$

Therefore, $f : \mathbb{R} \to \tilde{\mathbb{R}}$ is smooth if and only if $x \to f(x)^5$ is a smooth function on $\mathbb{R}$.

Now, choose f to be φ^{-1}, that is, $f(x) = \sqrt[5]{x}$.Then the local expression of f becomes $\bar{f}(x) = x$, so $\bar{f}$ is the identity map on $\mathbb{R}$ and hence it is smooth. Therefore, f is smooth. On the other hand, the inverse f^{-1} has the same local expression as f, namely equal to the identity on $\mathbb{R}$, so $f^{-1} : \tilde{\mathbb{R}} \to \mathbb{R}$ is also smooth. Thus we conclude that f is a diffeomorphism.

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Problem 3

Consider the maps

$$\psi : \Omega \to \mathbb{R}^3, \quad \psi(u, v) = (u, 1 - \cos v, \sin v)$$
$$\eta : \Delta \to \mathbb{R}^2, \quad \eta(x, y, z) = (x, \arccos(1 - y))$$

where $\Omega = \{(u, v) \in \mathbb{R}^2; \, 0 < v < \pi)$ and $\Delta = \{(x, y, z) \in \mathbb{R}^3; \, 0 < y < 2\}$.

a) Show that the Jacobi matrix $D_p\psi$ at each point $p = (u, v)$ belongs to the matrix space

$$O(3, 2) = \{A \in \mathbb{R}^{3\times 2}; A^T A = I\}$$

where $\mathbb{R}^{3\times 2}$ denotes the set of 3×2-matrices. Use this property of ψ to show that the tangent map at p

$$\psi_* : \mathbb{R}^2 = T_p\Omega \to T_{\psi(p)}\mathbb{R}^3 = \mathbb{R}^3$$

preserves the Euclidean inner product of tangent vectors, that is,

$$\langle \psi_*(\mathbf{a}), \psi_*(\mathbf{b}) \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$$

Solution The Jacobi matrix is

$$D\psi = \begin{pmatrix} 1 & 0 \\ 0 & \sin v \\ 0 & \cos v \end{pmatrix}$$

and

$$(D\psi)^T D\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin v & \cos v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sin v \\ 0 & \cos v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $D\psi$ belongs to $O(3,2)$.

The tangent spaces of $\Omega \subset \mathbb{R}^2$ and $\mathbb{R}^3$ at any point are naturally identified with $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, and thus the usual inner product (dot product) on $\mathbb{R}^2$ and $\mathbb{R}^3$ becomes an inner product on the tangent spaces. With respect to the standard bases, the matrix of ψ_* is precisely the matrix $D\psi$. Moreover, if vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are represented as column matrices, e.g., $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = a_1 b_1 + a_2 b_2$, then

$$\psi_*(\mathbf{a}) = (D\psi)\mathbf{a} = (D\psi) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and

$$\begin{aligned} \langle \psi_*(\mathbf{a}), \psi_*(\mathbf{b}) \rangle &= \langle (D\psi)\mathbf{a}, (D\psi)\mathbf{b} \rangle = ((D\psi)\mathbf{a})^T (D\psi)\mathbf{b} \\ &= \mathbf{a}^T (D\psi)^T (D\psi)\mathbf{b} = \mathbf{a}^T \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle \end{aligned}$$

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b) Show that ψ is an injective immersion. Moreover, show that the image surface $M = \psi(\Omega)$ is a submanifold of $\mathbb{R}^3$ which is diffeomorphic to Ω. (Hint: Consider the restriction $\eta|_M$ of the above map η.)

Solution For ψ to be an immersion, it must have rank 2 at each point. But the rank of ψ is the rank of the matrix $D\psi$, which is always equal to 2.

To show that ψ is also globally injective, assume $\psi(u_1, v_1) = \psi(u_2, v_2)$. Clearly, $u_1 = u_2$, but also $\cos v_1 = \cos v_2$. Since we assume v lies in the interval $(0, \pi)$, where $\cos v$ is strictly decreasing, it follows that $v_1 = v_2$. We conclude that ψ is injective.

The image of ψ is the surface $M = \psi(\Omega)$ in $\mathbb{R}^3$. We give M the induced topology from $\mathbb{R}^3$, so we know from the above that

$$\psi : \Omega \to M$$

is a continuous and bijective map. In order to establish that the map is also a diffeomorphism, we must ensure the following:

(i) $\psi^{-1} : M \to \Omega$ is continuous, (ii) M is a smooth manifold, and
(iii) $\psi^{-1} : M \to \Omega$ is smooth

In fact, a homeomorphism which is smooth is also a diffeomorphism, by the inverse function theorem, which guarantees that the inverse will necessarily be smooth.

It is perhaps surprising that the properties (ii) and (iii) follow from property (i). In fact, (i) says that the above map $\Omega \to M$ is a homeomorphism. Then it will follow from the definition of "immersion" and "submanifold" that M has the structure of a submanifold of $\mathbb{R}^3$ and, moreover, the map $\Omega \to M$ will be smooth because $\Omega \to \mathbb{R}^3$ is smooth and $M \subset \mathbb{R}^3$ is a submanifold.

So, the student need only verify that $\psi^{-1} : M \to \Omega$ is continuous – this is the essense of the question asked. Now, a simple calculation shows that the composition $\eta \circ \psi$ is the indentity map. In particular, ψ^{-1} is the restriction of η to M. But η is a smooth map on an open set containing M, so clearly the restriction $\eta|M$ is continuous.

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c) As submanifolds of Euclidean spaces Ω and M have induced metric structures. We claim that the map $\psi : \Omega \to M$ is also an isometry. This is a consequence of any one of the following two equivalent statements:

(i) The length of any smooth curve in Ω is equal to the length of its image curve in M.

(ii) Along M the coordinates x, y, z in $\mathbb{R}^3$ are functions of u and v. The following identity holds :

$$du^2 + dv^2 = (dx^2 + dy^2 + dz^2)|_M.$$

Choose either (i) or (ii) and prove the statement.

Solution (i) We show that the length of a smoothly parametrized curve $t \to \gamma(t)$ in Ω, $t \in [a, b]$, will be mapped to a curve $\psi \circ \gamma$ on M having the same length. The velocity of the curve at $p = \gamma(t)$ is the tangent vector $\gamma'(t) = \frac{d}{dt}\gamma(t)$ in the tangent plane $T_p\Omega = \mathbb{R}^2$, and the length L of a curve is obtained by integrating the speed, hence

$$L(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt = \int_a^b \sqrt{\langle \psi_*(\gamma'(t)), \psi_*(\gamma'(t)) \rangle} dt$$

$$= \int_a^b \sqrt{\langle (\psi \circ \gamma)'(t)), (\psi \circ \gamma)'(t)) \rangle} dt = L(\psi \circ \gamma)$$

(ii) The Euclidean metric on $\mathbb{R}^2$, with (u, v) regarded as coordinates with respect to an orthonormal basis, can be expressed by the equation

$$ds^2 = du^2 + dv^2$$

where ds is the arc-length element. The expression represents the Riemannian metric in terms of the coordinates (u, v). Similarly, in Euclidean 3-space with coordinates (x, y, z) with respect to an orthonormal basis, the (Riemannian) metric expresses as

$$ds^2 = dx^2 + dy^2 + dz^2$$

Then there will be induced a (Riemannian) metric on any submanifold of $\mathbb{R}^3$. Let us calculate the induced metric on M, in terms of the coordinates (u, v)

on M defined by the global chart (ψ^{-1}, M). To do this, regard the coordinates x, y, z restricted to M as functions of u and v and calculate their differential :

$$dx = du, dy = \sin v dv, dz = \cos v dv$$

Then $dx^2 = du^2, dy^2 = \sin^2 v dv^2, dz^2 = \cos^2 v dv^2$, and their sum equals $du^2 + dv^2$.This proves the statement (ii).

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Remark The above map $\psi : \Omega \to \mathbb{R}^3$ is a *rigid map* which provides an example of an *origami,* the Japanese art of folding paper. Namely, Ω represents a thin sheet of paper which is immersed into 3-space according to the map ψ. The paper is rigid in the tangential directions, and it cannot be stretched, compressed, or sheared. The paper is allowed to be folded, but then ψ would not be smooth along the folding curve, although still continuous. In our case the paper sheet is rolled up to a sylinder perpendicular to the yz-plane direction. For the interesting reader, we refer to a recent article on origami with many pictures : "Origami and Partial Differential Equations", in Notices of the Amer. Math. Society, May 2010.

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Problem 4

In homogeneous coordinates, a point in the projective plane is represented by a nonzero triple $[x_1, x_2, x_3]$, with at least one $x_i \neq 0$, together with the rule that $[x_1, x_2, x_3] = [kx_1, kx_2, kx_3]$ whenever $k \neq 0$. The topology of P^2 is the quotient topology of $\mathbb{R}^3 - \{0\}$ (or S^2) via the map $\pi : (x_1, x_2, x_3) \to [x_1, x_2, x_3]$. We cover P^2 with the open sets

$$U_i = \{[x_1, x_2, x_3]; x_i \neq 0\}, \;\; i = 1, 2, 3.$$

Show that P^2 is a 2-dimensional smooth manifold by constructing chart functions $\varphi_i : U_i \to \mathbb{R}^2$ so that $\{(\varphi_i, U_i), i = 1, 2, 3\}$ is a smooth atlas. Is P^2 compact?

Solution Consider the projection map π :

$$S^2 \subset \mathbb{R}^3 - \{0\} \to P^2$$

P^2 has the quotient topology from $\mathbb{R}^3 - \{0\}$. For any point $[x_1, x_2, x_3]$ in P^2 one can, after muliplying the $x_i's$ with some number k, assume that $x_1^2 + x_2^2 + x_3^2 = 1$. Therefore, P^2 is also the image of S^2.In fact, precisely the two antipodal points (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ on S^2 are mapped to the same point $[x_1, x_2, x_3]$.

S^2 is both a Hausdorff and compact space, and being the image of a compact space via a continuous map, P^2 is also compact. It is also easy to see that P^2 is Hausdorff, by considering the map $S^2 \to P^2$, but we shall not pay any attention to this.

Finally, we turn to the construction of a smooth atlas $\{(\varphi_i, U_i), i = 1, 2, 3\}$ on P^2. We define the chart φ_3 as follows:

$$\varphi_3 : [x_1, x_2, x_3] = [\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1] \to (\frac{x_1}{x_3}, \frac{x_2}{x_3}).$$

Clearly it is a bijective map $U_3 \to \mathbb{R}^2$, and we claim it is a homeomorphism. Observe that the open set $\pi^{-1}(U_3)$ of S^2 consists of all (x_1, x_2, x_3) with $x_3 \neq 0$, which is union of the upper hemisphere ($x_3 > 0$) and the lower hemisphere ($x_3 < 0$), and π maps each hemisphere homeomorphically to U_3. We see that the composition $\varphi_3 \circ \pi$ is a homeomorphism $(x_1, x_2, x_3) \to (\frac{x_1}{x_3}, \frac{x_2}{x_3})$ from each of the hemispheres to $\mathbb{R}^2$. In particular, φ_3 is a homeomorphism. The charts φ_1 and φ_2 are defined similarly to φ_3, and they are seen to be homeomorphisms by similar reasoning.

It remains to show that all the transition functions $\varphi_i \circ \varphi_j^{-1}$ are smooth (on their domain of definition). Since all cases are similar, let us choose the following :

$$\varphi_2 \circ \varphi_1^{-1} : (x, y) \to [1, x, y] = [\frac{1}{x}, 1, \frac{y}{x}] \to (\frac{1}{x}, \frac{y}{x})$$

which is a diffeomorphism of the set $\{(x, y) \in \mathbb{R}^2; x \neq 0\}$ to itself.

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Problem 5

Let M be a 2-dimensional connected smooth manifold and

$$f : M \to \mathbb{R}$$

a smooth map which is surjective and has no critical points. Assume also f is proper, that is, the inverse image of a compact set is compact. Use Ehresmann's fibration theorem to show that M is a product $F \times B$ (modulo a diffeomorphism) of two manifolds. What manifolds are F and B?

Solution Since f has no critical points, it has rank 1 everywhere, so it is a submersion. According to Ehresmann's theorem $f : M \to \mathbb{R}$ is the projection of a locally trivial fibration (or fiber bundle). We indicate this as follows

$$F \to M \to \mathbb{R}$$

where F is the fiber, that is, the $f^{-1}(p) \simeq F$ for all points p in $\mathbb{R}$. Namely, all fibers are diffeomorphic to the same manifold F.

What can F be ? Since p is a regular value for f, the fiber is a 1-dimensional submanifold of M, and it is compact since f is proper. Therefore F must be a disjoint union of circles, say $F = S_1^1 \cup S_2^1 \cup ... \cup S_k^1$. On the other hand, because of the special simple topology of $\mathbb{R}$ (it is contractible) all fiber bundles (including vector bundles) over $\mathbb{R}$ must, in fact, be trivial. That is, the fiber bundle is a product bundle, so that $M \simeq F \times \mathbb{R}$. But we know that M is connected, and the product space cannot be connected unless $k = 1$, hence $M \simeq S^1 \times \mathbb{R}$.

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Note The student was not expected to be able to show that the above fibration is trivial. In fact, this is just the statement of Lemma 9.5.7 in Dundas, with $n = 1$. The general theory of fiber bundles belongs to algebraic topology.