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Exam in TMA4190 Manifolds

English
Wednesday June 2, 2010
Time : 09.00–13.00

Permitted aids : Code D
Grades: June 23, 2010

Instructions. In the problems below, the term differentiable (or smooth) map or manifold means C^∞-differentiable.

Problem 1

a) Consider the cone

$$C = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = z^2\}$$

with the topology induced as a subset from $\mathbb{R}^3$. Is C a topological manifold ? Explain.

b) Show that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi : (x, y) \to (xe^y + y, xe^y - y)$$

is a diffeomorphism.

Problem 2

a) Let M be a topological manifold. Explain briefly what is a smooth structure on M, and what is a smooth map $f : M \to M$.

b) Consider the homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x^5$. Let $\mathcal{A}$ denote the standard smooth structure of $\mathbb{R}$ and let $\mathcal{B}$ denote the smooth structure containing the chart $(\mathbb{R}, \varphi)$. Show that $\mathcal{A}$ is different from $\mathcal{B}$.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Describe the property f must have in order to be a smooth map $f : \mathbb{R} \to \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}$ denotes the real line $\mathbb{R}$ with the smooth structure $\mathcal{B}$. Find a map f which is a diffeomorphism between $\mathbb{R}$ and $\tilde{\mathbb{R}}$?

Problem 3

Consider the maps

$$\psi : \Omega \to \mathbb{R}^3, \quad \psi(u,v) = (u, 1-\cos v, \sin v)$$
$$\eta : \Delta \to \mathbb{R}^2, \quad \eta(x,y,z) = (x, \arccos(1-y))$$

where $\Omega = \{(u,v) \in \mathbb{R}^2;\ 0 < v < \pi\}$ and $\Delta = \{(x,y,z) \in \mathbb{R}^3;\ 0 < y < 2\}$.

a) Show that the Jacobi matrix $D_p\psi$ at each point $p = (u,v)$ belongs to the matrix space

$$O(3,2) = \{A \in \mathbb{R}^{3\times 2};\ A^T A = I\}$$

where $\mathbb{R}^{3\times 2}$ denotes the set of 3×2-matrices. Use this property of ψ to show that the tangent map at p

$$\psi_* : \mathbb{R}^2 = T_p\Omega \to T_{\psi(p)}\mathbb{R}^3 = \mathbb{R}^3$$

preserves the Euclidean inner product of tangent vectors, that is,

$$\langle \psi_*(\mathbf{a}), \psi_*(\mathbf{b}) \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$$

b) Show that ψ is an injective immersion. Moreover, show that the image surface $M = \psi(\Omega)$ is a submanifold of $\mathbb{R}^3$ which is diffeomorphic to Ω. (Hint: Consider the restriction $\eta|_M$ of the above map η.)

c) As submanifolds of Euclidean spaces Ω and M have induced metric structures. We claim that the map $\psi : \Omega \to M$ is also an isometry. This is a consequence of any one of the following two equivalent statements:

(i) The length of any smooth curve in Ω is equal to the length of its image curve in M.

(ii) Along M the coordinates x, y, z in $\mathbb{R}^3$ are functions of u and v. The following identity holds :

$$du^2 + dv^2 = (dx^2 + dy^2 + dz^2)|_M.$$

Choose either (i) or (ii) and prove the statement.

Problem 4

In homogeneous coordinates, a point in the projective plane is represented by a nonzero triple $[x_1, x_2, x_3]$, with at least one $x_i \neq 0$, together with the rule that $[x_1, x_2, x_3] = [kx_1, kx_2, kx_3]$ whenever $k \neq 0$. The topology of P^2 is the quotient topology of $\mathbb{R}^3 - \{0\}$ (or S^2) via the map $\pi : (x_1, x_2, x_3) \to [x_1, x_2, x_3]$. We cover P^2 with the open sets

$$U_i = \{[x_1, x_2, x_3];\ x_i \neq 0\}, \quad i = 1,2,3.$$

Show that P^2 is a 2-dimensional smooth manifold by constructing chart functions $\varphi_i : U_i \to \mathbb{R}^2$ so that $\{(\varphi_i, U_i), i = 1,2,3\}$ is a smooth atlas. Is P^2 compact?

Problem 5

Let M be a 2-dimensional connected smooth manifold and

$$f : M \to \mathbb{R}$$

a smooth map which is surjective and has no critical points. Assume also f is proper, that is, the inverse image of a compact set is compact. Use Ehresmann's fibration theorem to show that M is a product $F \times B$ (modulo a diffeomorphism) of two manifolds. What manifolds are F and B?