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Department of Mathematics

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English version

Manifolds (TMA4190): Final Exam

Saturday 6th June 2009
Time: 9:00–13:00
Permitted Examination Aids: D
No written or handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Give a definition of a smooth manifold.

For each part of your definition, give a short explanation of what it means and why it is included. *(10 points)*

Solution:

A smooth manifold consists of a locally Euclidean, metrisable topological space together with a choice of smooth structure.

Alternatives to metrisable: Hausdorff and paracompact, Hausdorff and second countable, Hausdorff and sigma compact.

Alternatives to smooth structure: maximal smooth atlas.

A topological space is *locally Euclidean* if every point has an open neighbourhood homeomorphic to (an open subset of) some Euclidean space.

A topological space is *metrisable* if it can be given a metric which generates its topology.

A smooth structure on a *topological manifold* (which is what a smooth manifold without a smooth structure is called) is a maximal smooth atlas. A smooth atlas is a cover of open sets together with homeomorphisms to open subsets of Euclidean spaces such that the transition functions are diffeomorphisms. It is maximal if it is not contained in a strictly larger smooth atlas.

locally Euclidean this is to ensure that we have the local structure needed to make sense of questions about differentiability

metrisable this is to ensure that the topology is reasonable

smooth structure this is what enables us to discuss differentiability consistently

Problem 2.

Let $S^2 = \{(x, y, z) \subseteq \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Describe the standard smooth manifold structure on S^2 and show that it really is a smooth structure on S^2. *(10 points)*

Solution:

To describe the standard smooth structure we first describe a smooth atlas. There are many that would do, let us take stereographic projection from the poles. Define

$$\psi_N \colon \mathbb{R}^2 \to S^2 \setminus \{N\}, \qquad \psi_N(u, v) = \left(\frac{4u}{4 + u^2 + v^2}, \frac{4v}{4 + u^2 + v^2}, \frac{4 - (u^2 + v^2)}{4 + u^2 + v^2} \right)$$

this is invertible with inverse

$$\psi_N^{-1} \colon S^2 \setminus \{N\} \to \mathbb{R}^2, \qquad \psi_N^{-1}(x, y, z) = \left(\frac{2x}{1 - z}, \frac{2y}{1 - z} \right).$$

Both of these are continuous as maps from or to (open subsets of) $\mathbb{R}^3$ and hence are continuous as maps to or from $S^2 \setminus \{N\}$. Hence ψ_N is a homeomorphism.

Define ψ_S similarly. The transition functions in both directions are the map

$$\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \qquad (u, v) \mapsto \frac{4}{u^2 + v^2}(u, v)$$

which is smooth, and hence a diffeomorphism.

Thus $\{\psi_N, \psi_S\}$ defines a smooth atlas for S^2. We now take the unique maximal smooth atlas containing these two charts. That is, we define

$$\mathcal{A} = \{\phi \colon \mathbb{R}^2 \supseteq U \to V \subseteq S^2 : \psi_{S/N}^{-1}\phi \text{ is a diffeomorphism}\}$$

Problem 3.

The following is a list of properties that a smooth map between manifolds can have.

i. diffeomorphism
ii. embedding
iii. fibre bundle (i.e. the map in question is the projection map of a fibre bundle)
iv. homeomorphism
v. immersion
vi. submersion

Give a list of which properties imply which other properties. Give examples of functions which show that the reverse implications are not true.

As an example, a function which is a diffeomorphism is automatically a local diffeomorphism so the property of being a diffeomorphism implies that of being a local diffeomorphism. However, the reverse implication is false as evidenced by the function $\mathbb{R} \to S^1$, $t \mapsto (\cos t, \sin t)$ which is a local diffeomorphism but not a diffeomorphism.

(10 points)

Solution:

The implications are:

- diffeomorphism implies all the others
- embedding implies immersion
- fibre bundle implies submersion

Possible counterexamples are:

- A homeomorphism that is not a diffeomorphism: $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$.
- An embedding that is not a diffeomorphism: $S^1 \to \mathbb{R}^2$, $(x, y) \mapsto (x, y)$.
- An immersion that is not an embedding: $S^1 \to S^1$, $(\cos\theta, \sin\theta) \mapsto (\cos 2\theta, \sin 2\theta)$; this is also an immersion that is not a diffeomorphism.
- A fibre bundle that is not a diffeomorphism: $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$.

Problem 4.

i. For each $a \in \mathbb{R}$, define a map $f_a \colon \mathbb{R} \to \mathbb{R}^2$ by

$$f_a(t) = (t^2, t(t^2 - a)).$$

For which values of a is this an immersion and for which values of a is this an embedding? *(6 points)*

ii. For $a \in \mathbb{R}$, define a map $g_a \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g_a(x, y) = y^2 - x^3 - ax.$$

Find the critical values of g_a. *(4 points)*

Solution:

i. As an embedding is a special kind of immersion, we start by checking the condition for an immersion. To do this, we compute the derivative of the map. This is

$$Df_a(t) = \begin{bmatrix} 2t \\ 3t^2 - a \end{bmatrix}$$

As the domain is $\mathbb{R}$, for this to fail to be injective it must be null. For this to happen we must have $t = 0$ for the first entry and $3t^2 = a$ for the second. Thus if $a = 0$ it fails to be injective at $t = 0$ but for $a \neq 0$ it is always injective. Hence f_a is an immersion for $a \neq 0$.

For an embedding, we note first that an embedding must be injective. For $a > 0$ then when $t^2 = a$ we have $f_r(t) = (a, 0)$. As there are two points $(\pm\sqrt{a})$ where $t^2 = a$ the map fails to be injective.

For $a < 0$ consider the projection onto the y–axis, $t \mapsto t(t^2 - a)$. As a is negative, the derivative of this is strictly positive. It is therefore a diffeomorphism (onto its image) by the inverse function theorm. Hence f_a has a continuous inverse and so is a homeomorphism onto its image. Thus for $a < 0$, f_a is an embedding.

ii. To find the critical values we differentiate the map. This yields

$$Dg_a(x, y) = \begin{bmatrix} 3x^2 - a & 2y \end{bmatrix}$$

This has rank either 0 or 1. The places where it has rank 0 are when it vanishes completely. This means that $y = 0$ and $3x^2 = a$. Thus if $a < 0$ there are no critical points and hence no critical values.

If $a \geq 0$ then the rank is 0 at $(\pm\sqrt{a/3}, 0)$. This is a two-point set and so its complement is open. Thus all other points are regular. The values of g_a at these points are $\pm 4(a/3)^{3/2}$. Hence these are the critical values of g_a.