Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
Idar Hansen. Phone: 73 59 35 28.

Exam in TMA4190 Manifolds

English.
Tuesday June 3, 2008
Time: 09.00 – 13.00

Permitted aids: Code D.
Grades: June 23, 2008

Problem 1

a) Let X be a topological space. Give the definintion of the closure $\overline{A}$ of a subset $A \subseteq X$. Show that $\overline{A}$ is a closed subset of X.

b) Let X and Y be topological spaces. Show that if $f\colon X \to Y$ is continuous, then $f(\overline{A}) \subseteq \overline{f(A)}$.

c) Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$ be two topologies on the set X. Show that if X is compact in the topology $\mathcal{T}_1$ and Hausdorff in the topology $\mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 2 Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ be bilinear (i.e. f is linear in each variable), and suppose $\|f(x,y)\| \leq \|x\|\,\|y\|$ for all $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$. Show that f is differentiable, and that

$$Df(x,y)(h,k) = f(x,k) + f(h,y)$$

for all $(x,y), (h,k) \in \mathbb{R}^m \times \mathbb{R}^n$.

Problem 3

a) Give the definition of a (smooth) submanifold of a smooth manifold.

b) Show that if N is submanifold of M, then TN is a submanifold of TM.

c) Let M and N be smooth manifolds, and let $f\colon M \to N$ be a smooth map. Show that $f^{-1}(q) \subseteq M$ is a submanifold if $q \in f(M) \subseteq N$ is a regular value.

Problem 4 Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f\colon U \to \mathbb{R}^m$ be a smooth function with $f(0) = 0$. Assume f has rank m at the origin. Show that there exists a chart (V, φ) at 0 in $\mathbb{R}^n$ with $\varphi(V) \subseteq U$ such that for all $(x,y) \in V \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$ we have $f \circ \varphi(x,y) = x$.

Hint: Permute the coordinates if necessary, and consider the function $\tilde{f} : U \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ given by $\tilde{f}(x,y) = (f(x,y),y)$.

Problem 5 In this problem we identify the tangent space $T_A GL(n,\mathbb{R})$ with the vector space $\mathbb{R}^{n\times n}$ of $n \times n$-matrices by mapping $[\alpha]$ to $\alpha'(0)$.

Define $f : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ by $f(A) = \mathrm{tr}(A)A^{-1}$. Here $\mathrm{tr}(A)$ denotes the trace of A given by $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$ if $A = [a_{ij}]$.

a) Explain why f is differentiable, and find the derivative $d_A f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ at $A \in GL(n,\mathbb{R})$.

b) Show that if $\mathrm{tr}(A) = 0$, then A is not a regular point of f.