**Norwegian University of Science and Technology**
**Department of Mathematical Sciences**



Contact during the exam:
Idar Hansen 73 59 35 28

Exam in TMA4190 Manifolds
English
Friday June 1, 2007
Time: 09.00-13.00

Permitted aids: Code D

Grades: June 21, 2007

## Problem 1

Let $X$ and $Y$ be topological spaces, and let $f\colon X \to Y$ be a continuous function. Show that if $X$ is compact and $Y$ is Hausdorff, then $f$ is closed, i.e. $A$ closed in $X$ implies $f(A)$ closed in $Y$.

## Problem 2

Let $X$ have the finite complement topology. Show that if $X$ is infinite, then any two non-empty open sets have a non-empty intersection. What is $\overline{U}$ if $U \subseteq X$ is open and non-empty?

## Problem 3

Let $M$ and $N$ be smooth manifolds, and let $f\colon M \to N$ be a smooth function. Give the definition of the derivative $df\colon TM \to TN$ of $f$, and show that $df$ is smooth.

**Problem 4**

**a)** Define immersion and imbedding.

**b)** Let $f\colon M \to N$ be an injective immersion. Show that if $f$ is closed (see Problem 1), then $f$ is an imbedding.

**c)** Let $f\colon M \to N$ be a bijective immersion. Show that if $M$ and $N$ have the same dimension, then $f$ is a diffeomorphism.

**Problem 5**

Let $U \subseteq \mathbb{R}^n$ be an open set containing the origin, and let $f\colon U \to \mathbb{R}^m$ be a smooth function with $f(0) = 0$. Assume that $f$ has rank $n$ at the origin. Show that there exists a chart $(W, \psi)$ at $0$ in $\mathbb{R}^m$ and an open neighborhood $V$ of $0$ in $\mathbb{R}^n$ such that for all $x \in V$ we have $\psi \circ f(x) = (x, 0)$ in $\mathbb{R}^n \times \mathbb{R}^{m-n}$.

Hint: Permute coordinates if necessary, and consider the function $\tilde{f}\colon U \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ given by $\tilde{f}(x, y) = f(x) + (0, y)$.

**Problem 6**

Let $\mathbb{R}^{n\times n}$ denote the $n^2$-manifold (diffeomorphic to $\mathbb{R}^{n^2}$) of $n \times n$-matrices, and identify the tangent space $T_A\mathbb{R}^{n\times n}$ at $A$ with $\mathbb{R}^{n\times n}$ by mapping $[\alpha]$ to $\alpha'(0)$. Let $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be given by $f(A) = A^2$. Find the derivative $d_Af\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ of $f$ at $A \in \mathbb{R}^{n\times n}$.