Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam:
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Exam in TMA4190 MANIFOLDS
English
Tuesday June 6, 2006
Time: 09.00-13.00

Permitted aids: Code D

Grades: June 27, 2006

Problem 1

a) What does it mean for a topological space to be Hausdorff?

b) Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) | x \in X\} \subseteq X \times X$$

is closed.

c) Let X and Y be topological spaces, and let $f \colon X \to Y$ be a continuous function. Show that if Y is Hausdorff, then the graph $\Gamma_f = \{(x, f(x)) | x \in X\} \subseteq X \times Y$ is closed.

Problem 2

a) Give the definition of a submanifold of a smooth manifold.

Let M and N be smooth manifolds, and let $f : M \to N$ be a smooth map.

b) What is a regular point of f, and what is a regular value of f?

c) Let $q \in f(M)$ be a regular value of f. Show that $f^{-1}(q) \subseteq M$ is a submanifold.

d) Use c) to show that $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a submanifold.

Problem 3

a) Define imbedding and immersion.

b) Let the map $f\colon (-\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R}^2$ be given by $f(t) = (\sin 2t, \cos t)$. Show that f is an immersion. Is f an imbedding?

Problem 4 In this problem we identify the tangent space $T_A\mathrm{GL}(n,\mathbb{R})$ with the set of $n \times n$-matrices by mapping $[\alpha]$ to $\alpha'(0)$.

a) Show that the determinant $\det\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ is differentiable and has differential $d_A \det\colon T_A\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ given by $(d_A \det)(B) = \det A \,\mathrm{tr}(A^{-1}B)$. Here the trace is given by $\mathrm{tr}A = \sum_{i=1}^{n} a_{ii}$.

b) Show that the function $i\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ given by $i(A) = A^{-1}$ is differentiable, and find $d_A i\colon T_A\mathrm{GL}(n,\mathbb{R}) \to T_{i(A)}\mathrm{GL}(n,\mathbb{R})$.