TMA4190 Manifolds

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Solutions
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Problem 1

a) A topological space $(X, \mathscr{T})$ is compact if for any open cover $\{U_\alpha\}_\alpha$ of X, i.e., $X = \bigcup_\alpha U_\alpha$, we can find $\alpha_1, \dots, \alpha_n$ such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$.

b) Let $\{U_\alpha\}_\alpha$ be open in Y such that $f(X) \subseteq \bigcup_\alpha U_\alpha$. Then $X = \bigcup_\alpha f^{-1}(U_\alpha)$ and $\{f^{-1}(U_\alpha)\}_\alpha$ is an open cover of X. Since X is compact, $X = f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n})$ for some $\alpha_1, \dots, \alpha_n$, and hence $f(X) \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. Thus $f(X) \subseteq Y$ is compact.

c) Suppose $\bigcap F_n = \varnothing$ and let $U_n = F_n^c$. Then $U_1 \subseteq U_2 \subseteq \dots$ are open sets with $\bigcup U_n = \bigcup F_n^c = (\bigcap F_n)^c = \varnothing^c = X$. Thus $X = U_1 \cup \dots \cup U_n = U_n$ for some n. But then $F_n = U_n^c = X^c = \varnothing$, and we have a contradiction. Hence $\bigcap F_n \neq \varnothing$.

Problem 2

a) A smooth manifold is a pair $(M, \mathscr{U})$ where M is a second countable Hausdorff space, and $\mathscr{U}$ is a maximal smooth atlas on M. I.e., for any $(U, \varphi), (V, \psi)$ in $\mathscr{U}$, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth, and if $(U, \varphi) \in \mathscr{U}$ and (V, ψ) is any topological chart such that $\psi \circ \varphi^{-1}$ is smooth, then $(V, \psi) \in \mathscr{U}$. [Maximality.] Here (U, φ) is a topological chart if $U \subseteq M$ is open and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism with an open subset of $\mathbb{R}^n$.

b) A map $f\colon M \to N$ is smooth if f is continuous and for any $(V, \psi) \in \mathscr{V}$ and $(U, \varphi) \in \mathscr{U}$ the map $\psi \circ f \circ \varphi^{-1}\colon \varphi(U \cup f^{-1}(V)) \to \psi(V)$ is smooth.

c) Let $\mathscr{U}$ be an atlas for M. Then for each $(U, \varphi) \in \mathscr{U}$ we take the charts

$$\tilde{\varphi} : \varphi^{-1}(U) \to U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

given by $\tilde{\varphi}([\alpha]) = (\varphi(\alpha(0)), D(\varphi \circ \alpha)(0))$. This gives the special charts $(\varphi^{-1}(U), \tilde{\varphi})$ for TM. Here α is a smooth curve in U.

d) Consider

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \pi\ } & U \\ {\scriptstyle \tilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \varphi(U) \times \mathbb{R}^n & \xrightarrow{\ \hat{\pi}\ } & \varphi(U) \end{array}$$

(where $\hat{\pi} = \varphi \circ \pi \circ \tilde{\varphi}^{-1}$.) Then $\hat{\pi} = \pi_1$ since if $(x, v) = \tilde{\varphi}([\alpha])$, i.e., $\varphi \circ \alpha(0) = x$ and $D(\varphi \circ \alpha)(0) = v$, then $\hat{\pi}(x, v) = \hat{\pi} \circ \tilde{\varphi}([\alpha]) = \varphi \circ \pi([\alpha]) = \varphi \circ \alpha(0) = x$. Since $\hat{\pi}$ is smooth, so is π (π is continuous).

Problem 3

a) Define a smooth chart on $\mathbb{R}^n \times \mathbb{R}^m$ by $\varphi(x,y) = (x, y - f(x))$. (Then φ is smooth and $\varphi^{-1}(u,v) = (u, v+f(u))$ shows that φ^{-1} is smooth.) Then $\varphi((\mathbb{R}^n \times \mathbb{R}^m) \cap \Gamma_f) = \varphi(\Gamma_f) = \mathbb{R}^n \times \{0\}$, and Γ_f is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$.

b) Fix $y \in f(A) \subseteq \mathbb{R}^m$, and let $B = f^{-1}(y) \subseteq A$. Then $B \subseteq A$ is a closed subset, and $B \neq \emptyset$ since $y \in f(A)$. We show that B also is open. Let $x \in B$ and let N be an open ball about x in A. For any $x' \in N$ (since N is convex) we have $\|f(x') - f(x)\| \leq M\|x' - x\|$ where $M = \max\limits_{z \in [x',x]} \|Df(z)\| = 0$. Thus $f(x') = f(x) = y$. Hence $N \subseteq B$ and B is open. Then $A = B \cup B^c$ ($B^c = A - B$) with both B and B^c both open and closed. Since $B \neq \emptyset$, we have $B^c = \emptyset$, thus $B = A$ and f is constant.

Problem 4

a) If $f\colon M \to N$ is a diffeomorphism, then $d_pf\colon T_pM \to T_{f(p)}N$ is an isomorphism, and hence $\dim M = \dim T_pM = \dim T_{f(p)}N = \dim N$.

b) Let $\dim M = m$ and $\dim N = n$, and suppose f has constant rank k. If it is not an immersion, then $k < m$. By the Rank Theorem we can for each $p \in M$ find charts (U,φ) and (V,ψ) such that $\psi \circ f \circ \varphi^{-1}(x_1,\ldots,x_k,\ldots,x_m) = (x_1,\ldots,x_k,0,\ldots,0)$ near p. For $x_{k+1},\ldots,x_m$ sufficiently small, $\psi \circ f \circ \varphi^{-1}(0,\ldots,0,x_{k+1},\ldots,x_m) = (0,\ldots,0)$ and f is not injective. Hence f is an immersion.

c) Define $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{Sym}(n)$ by $f(A) = A^TA$. Then f is smooth and $f^{-1}(I) = \mathrm{O}(n)$. If we show that I is a regular value, then $\mathrm{O}(n) \subseteq \mathrm{GL}(n,\mathbb{R})$ is a submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. Consider

$$\begin{array}{ccc} T_A\mathrm{GL}(n,\mathbb{R}) & \xrightarrow{d_Af} & T_{f(A)}\mathrm{Sym}(n) \\ \cong\big\downarrow\mu & & \cong\big\downarrow\mu \\ \mathbb{R}^{n\times n} & \xrightarrow{D_Af} & \mathrm{Sym}(n) \end{array}$$

where $\mu([\alpha]) = \alpha'(0)$ ($\alpha(0) = A$ or $f(A)$). Let $\alpha(0) = A, \alpha'(0) = B$, then

$$\begin{aligned} (D_Af)B &= (D_Af)\alpha'(0) \\ &= \mu \circ d_Af([\alpha]) \\ &= \mu([f \circ \alpha]) \\ &= (f \circ \alpha)'(0) \\ &\vdots \\ &= B^TA + A^TB \end{aligned}$$

since $(f \circ \alpha)(t) = \alpha(t)^T\alpha(t)$ gives that $(f \circ \alpha)'(t) = \alpha'(t)^T\alpha(t) + \alpha(t)^T\alpha'(t)$.

If $A \in \mathrm{O}(n)$, then $D_A f$ is onto since given $C \in \mathrm{Sym}(n)$

$$\begin{aligned}(D_A f)(\frac{1}{2}AC) &= \frac{1}{2}(C^T A^T A + A^T AC) \\ &= \frac{1}{2}(C^T + C) = C.\end{aligned}$$

Thus I is a regular value and we are done.