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Contact during the exam:
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EXAM IN COURSE TMA4190 MANIFOLDS

English
Tuesday, May 24, 2005
Time 9–13

Permitted aids: (Code D)

Grades: June 14, 2005.

Problem 1

a) What is a compact topological space?

b) Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous function. Show that if X is compact, then $f(X)$ is a compact subspace of Y.

c) Let X be a compact topological space, and suppose $\{F_n | n = 1, 2, \cdots\}$ is a family of non-empty, closed subsets of X such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$. Show that $\cap_{n=1}^{\infty} F_n \neq \varnothing$.

Problem 2

a) Give the definition of a smooth n-manifold.

b) Let M and N be smooth manifolds. Give the definition of a smooth map $f\colon M \to N$.

c) The tangent space TM of a smooth n-manifold M is a smooth $2n$-manifold. Coming from charts on M we have special charts on TM showing this. What are these charts?

d) Show that the map $\pi\colon TM \to M$ given by $\pi([\alpha]) = \alpha(0)$ is smooth.

Problem 3

a) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, then the graph Γ_f of f is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$. Here $\Gamma_f = \{(x, f(x)) | x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$.

b) Let $A \subseteq \mathbb{R}^n$ be a non-empty, open and connected set, and let $f\colon A \to \mathbb{R}^m$ be a smooth funcion with $Df(x) = 0$ for all $x \in A$. Show that f is a constant function.

Problem 4

a) Show that if two smooth manifolds are diffeomorphic, then they have the same dimension.

b) Let M and N be smooth manifolds, and let $f\colon M \to N$ be a smooth map of constant rank. Show that if f is injective (one-to-one), then f is an immersion.

c) Show that the othogonal group $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ is a submanifold of $\mathrm{GL}(n, \mathbb{R})$ of dimension $\frac{n(n-1)}{2}$.