Lecture 3

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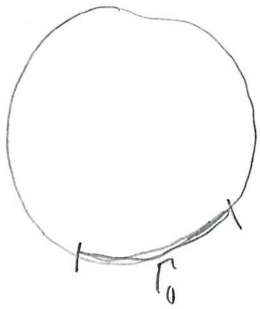
2. Linear elliptic PDEs

2.1 Variational formulation

Let $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary



then we consider

$$
(SE) \begin{cases} -\operatorname{div}(k(x)\operatorname{grad} y) = f & \text{in } \Omega \quad (1) \\ y = 0, & \text{on } \Gamma_0 \quad (2) \\ -k \dfrac{\partial y}{\partial \nu} = \beta (y - y_a) & \text{on } \Gamma_1 \quad (3) \end{cases}
$$

A. Derive variational formulation

multiply (1) with test function $\varphi \in H^1_{\Gamma_0}(\Omega)$

where $H^1_{\Gamma_0} = \{ v \in H^1(\Omega) \mid v|_{\Gamma_0} = 0 \} \subset H^1(\Omega)$

apply (formally) Green's formula

Notation:

(2) is called Dirichlet or essential cond.

(3) " " bound. cond. of 3^{rd} type

or Newton cooling law

(1)+(3) is called of mixed type

$-k \frac{\partial y}{\partial \nu} = g$ is called Neumann or

natural BC

Now, apply Green's formula:

$$- \int_\Omega \operatorname{div}(k(x) \operatorname{grad} y)\, \varphi \, dx = - \sum_{k=1}^{n} \int_\Omega \frac{\partial}{\partial x_k}\left(k(x) \frac{\partial y}{\partial x_k}\right) \varphi \, dx$$

$$= + \sum_{k=1}^{n} \left[\int_\Omega k(x) \frac{\partial y}{\partial x_k} \frac{\partial \varphi}{\partial x_k} dx - \int_{\Gamma_1} k(s) \frac{\partial y}{\partial x_k} \nu_k(s) \varphi(s)\, ds \right]$$

$$= \int_\Omega k(x) \nabla y \cdot \nabla \varphi \, dx - \underbrace{\int_{\Gamma_1} k(s) \frac{\partial y}{\partial \nu} \varphi(s)\, ds}_{= \beta \int_{\Gamma_1} (y - y_a) \varphi \, ds}$$

all in all, we get

$$(WF) \quad \underbrace{\int_\Omega k(x)\nabla y \cdot \nabla \varphi \, dx + \beta \int_{\Gamma_1} y \varphi \, ds}_{= a(y,\varphi)} = \underbrace{\int_\Omega f \varphi \, dx + \beta \int_{\Gamma_1} y_a \varphi \, ds}_{=: F(\varphi)}$$

i.e., (SE) $\Rightarrow$

Variational Formulation

$$a(y,\varphi) = F(\varphi) \quad \forall \varphi \in H^1_{\Gamma_0}(\Omega)$$

Assumptions on data

i) $f \in L^2(\Omega)$

ii) $y_a \in L^2(\partial\Omega)$

iii) $k \in L^\infty(\Omega)$, s.th. $0 < k_0 \leq k(x) \leq k_1 < \infty$ a.e. in Ω

iv) $\beta > 0$ and $|\Gamma_1| > 0$ or $\beta = 0$ and $\Gamma_0 = \partial\Omega$

Remarks:

1. (WF) is called weak or variational formulation
2. (SE) is the strong formulation
3. If $y \in H^2_{\Gamma_0}(\Omega)$ sol. to (SE) $\Rightarrow$ y sol. to (WF)

   for (WF) $\Rightarrow$ (SE) further assumptions needed