

Rothe method for parabolic equations

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- assume $\Omega \in \mathbb{R}^n$ with smooth boundary (at least $C^{1,1}$)
- consider

$$y_t - \operatorname{div}(k(x) \operatorname{grad} y) = \alpha u \quad \text{in } \Omega \times (0, T). \quad (1)$$

$$\frac{\partial y}{\partial \nu} = 0 \quad \text{in } \partial\Omega \times (0, T). \quad (2)$$

$$y(0) = y_0 \quad \text{in } \Omega. \quad (3)$$

- $T < \infty$ end-time
- we understand (1)–(3) as an evolution equation in a Hilbert space
- Definition

Let $I \subset \mathbb{R}$ an interval and H Hilbert space, then $C(I, H)$ is the space of all continuous functions $f : I \rightarrow H, t \mapsto f(t) \in H$ with norm

$$\|u\| = \sup_{t \in I} \|u(t)\|_H.$$

- $f \in C(I, H)$ not necessarily continuous in space, consider e.g., $g \in C([0, T])$ and $u \in L^2(\Omega)$, then $f(t) = g(t)u$ not continuous in x .
- thorough introduction see, e.g., Zeidler: Nonlinear FA, Vol. IIa, Springer, short intro in Evan's PDE book
- general idea: analogous to introduction of $L^p(\Omega)$, introduce $L^p(I, H)$ as completion of $C(I, H)$ with respect to norm

$$\|f\|_{L^p(0,T;H)} = \left(\int_I \|f(t)\|_H^p dt \right)^{1/p}.$$

- (1) $C([0, T]; L^2(\Omega))$ with norm $\|f\| = \sup_{[0, T]} \left(\int_{\Omega} f(t)^2 dt \right)^{1/2}$
- (2) $L^2((0, T); L^2(\Omega))$ with norm $\|f\| = \left[\int_0^T \left(\int_{\Omega} f^2(x, t) dx \right) dt \right]^{1/2}$
- (3) $H^1((0, T); L^2(\Omega))$ with norm

$$\begin{aligned} \|f\| &= \left(\int_0^T \left(\|f\|_{L^2(\Omega)}^2 + \|f_t\|_{L^2(\Omega)}^2 \right) dt \right)^{1/2} \\ &= \left(\int_0^T \int_{\Omega} (f^2(x, t) + f_t^2(x, t)) dx dt \right)^{1/2}. \end{aligned}$$

- (4) $L^2((0, T); H^1(\Omega))$ with norm

$$\|f\| = \left[\int_0^T \int_{\Omega} (f^2(x, t) + |\nabla f(x, t)|^2) dx dt \right]^{1/2}.$$

- Weak formulation

$$\int_{\Omega} y_t(t) \varphi dx + k \underbrace{\int_{\Omega} \nabla y(t) \nabla \varphi dx}_{=a(y(t), \varphi)} = \underbrace{\int_{\Omega} \alpha u \varphi dx}_{=F(\varphi)}$$

or

$$(y_t(t), \varphi)_{L^2(\Omega)} + a(y(t), \varphi) = F(\varphi) \quad \text{for all } \varphi \in H^1(\Omega), t \in (0, T) \quad (4)$$

$$y(0) = y_0 \quad \text{in } \Omega. \quad (5)$$

- we will show, (4),(5) has a unique sol.

$$y \in H^{1,1}(Q) := H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

where $Q = \Omega \times (0, T)$ is the space-time cylinder

- **Lemma:** Let $y \in H^{1,1}(Q)$ then $y \in C([0, T]; L^2(\Omega))$.
- Hence initial condition (5) is well-defined

- two strategies to prove existence of a weak solution

- Galerkin ansatz

Replace $V = H^1(\Omega)$ by finite dimensional approximation V_m , put $y^m(t) = \sum_{i=1}^m g_i(t)\varphi_i$, where φ_i is a basis of V_m . For m fixed, y^m is given as the solution to an ODE, passing to the limit gives solution to (4)–(5), see, e.g. Evan's book.

- Rothe method (which we will follow)

replace $y_t(t)$ by difference quotient $\frac{y(t+h)-y(t)}{h}$.

- Lemma:

Let $f \in L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 < p \leq \infty$. There is a constant $C > 0$, s.t. for all $\Omega' \subset\subset \Omega$ and $h < \text{dist}(\Omega', \partial\Omega)$ there holds

$$D_i^h f := \frac{f(x + he_i) - f(x)}{h} \in L^p(\Omega') \quad \text{with} \quad \|D_i^h f\|_{L^p(\Omega')} \leq C.$$

Then, $u_{x_i} \in L^p(\Omega)$ and

$$\|u_{x_i}\|_{L^p(\Omega)} \leq C.$$

- Assumptions on data

(A1) $k > 0$ konstant

(A2) $\alpha \in L^\infty(Q)$ mit $Q = \Omega \times (0, T)$ und $\|\alpha\|_{L^\infty(Q)} \leq \bar{\alpha}$

(A3) $u \in L^2(Q)$, $y_0 \in H^1(\Omega)$.

- **Theorem:** Assume (A1)–(A3), then (4)–(5) has a unique solution

$y \in H^{1,1}(Q)$.

- Proof in 4 steps

- 1 uniqueness via apriori estimate
- 2 time discretization, solution of elliptic problems
- 3 discrete apriori estimate
- 4 passing to the limit

Let y be a sol. to (4)–(5). Insert $\varphi = y_t$ in (4) integrate with respect to time

$$\int_0^t \int_{\Omega} y_{\xi}^2 dx d\xi + \underbrace{k \int_0^t \int_{\Omega} \nabla y \nabla y_{\xi} dx d\xi}_{I_1} = \underbrace{\int_0^t \int_{\Omega} \alpha u y_{\xi} dx d\xi}_{I_2} .$$

for the integrals we obtain

$$\begin{aligned} I_1 &= \frac{k}{2} \int_0^t \int_{\Omega} \frac{d}{d\xi} |\nabla y|^2 dx d\xi = \frac{k}{2}, \int_{\Omega} \int_0^t \frac{d}{d\xi} |\nabla y|^2 d\xi dx \\ &= \frac{k}{2} \int_{\Omega} |\nabla y(t)|^2 dx - \frac{k}{2} \int_{\Omega} |\nabla y_0|^2 dx . \end{aligned}$$

$$I_2 \leq \frac{1}{2} \int_0^t \int_{\Omega} y_{\xi}^2 dx d\xi + \frac{\bar{\alpha}^2}{2} \int_0^t \int_{\Omega} u^2 dx ds .$$

Altogether, we get

$$\begin{aligned} \frac{1}{2} \int_0^t \int_{\Omega} y_{\xi}^2 dx d\xi + \frac{k}{2} \int_{\Omega} |\nabla y(t)|^2 dx \\ \leq \frac{\bar{\alpha}^2}{2} \int_0^t \int_{\Omega} u^s dx d\xi + \frac{k}{2} \int_{\Omega} |\nabla y_0|^2 dx. \end{aligned} \quad (6)$$

Remark: Functions $y \in H^1(0, T; L^2(\Omega))$ are absolutely continuous, i.e.

$$y(t) = y(0) + \int_0^t y_{\xi} d\xi \quad \text{f.ü. in } \Omega$$

Using Hölder inequality and $(a + b)^2 \leq 2a^2 + 2b^2$ we get

$$\int_{\Omega} y(t)^2 dx = \int_{\Omega} \left(y_0 + \int_0^t y_{\xi} d\xi \right)^2 dx \leq 2 \int_{\Omega} y_0^2 dx + 2T \int_0^t \int_{\Omega} y_{\xi}^2 dx d\xi \quad (7)$$

hence, from (6) we get

$$\|y\|_{L^{\infty}(0, T; H^1(\Omega))}^2 + \|y\|_{H^1(0, T; L^2(\Omega))}^2 \leq C \left(\|u\|_{L^2(0, T; L^2(\Omega))}^2 + \|y_0\|_{L^2(\Omega)}^2 \right).$$

Since (4) is linear, we can infer uniqueness.

- Let $M \in \mathbb{N}$ fixed and $h = \frac{T}{M}$, $y^0 := y_0$. For $m \in \{1, \dots, M\}$ we define

$$u_\alpha^m(x) = \frac{1}{h} \int_{(m-1)h}^{mh} \alpha(x, t) u(x, t) dt. \quad (8)$$

- We replace $y_t(mh)$ by $\delta_h y^m := \frac{1}{h} (y^m - y^{m-1})$ and consider an implicit time-discrete scheme

$$\int_{\Omega} \delta_h y^m \varphi dx + k \int_{\Omega} \nabla y^m \nabla \varphi dx = \int_{\Omega} u_\alpha^m \varphi dx \quad \text{for all } \varphi \in H^1(\Omega). \quad (9)$$

- applying Lax-Milgram), inductively we get unique sol. $y^m \in H^1(\Omega)$ for $m = 1, \dots, M$ of elliptic problems (9).

We derive time discrete counterpart of (6). Insert $\varphi = y^m - y^{m-1}$ to get

$$\int_{\Omega} \delta_h y^m (y^m - y^{m-1}) dx + k \int_{\Omega} \nabla y^m \nabla (y^m - y^{m-1}) dx = \int_{\Omega} u_{\alpha}^m (y^m - y^{m-1}) dx.$$

Summing up for $m = 1$ to l yields

$$\sum_{m=1}^l h \int_{\Omega} (\delta_h y^m)^2 dx + \underbrace{k \sum_{m=1}^l \int_{\Omega} \nabla y^m \nabla (y^m - y^{m-1}) dx}_{I_2} = \underbrace{\sum_{m=1}^l \int_{\Omega} u_{\alpha}^m (y^m - y^{m-1}) dx}_{I_1}$$

We apply inequalities of Hölder und Young to I_1 ::

$$\begin{aligned} I_1 &\leq \sum_{m=1}^l h \int_{\Omega} u_{\alpha}^m \delta_h y^m dx \leq \frac{1}{2} \sum_{m=1}^l h \int_{\Omega} (u_{\alpha}^m)^2 dx + \frac{1}{2} \sum_{m=1}^l h \int_{\Omega} (\delta_h y^m)^2 dx \\ &\leq \frac{\bar{\alpha}^2}{2} \int_0^T \int_{\Omega} u^2 dx dt + \frac{1}{2} \sum_{m=1}^l h \int_{\Omega} (\delta_h y^m)^2 dx \end{aligned}$$

Here, we have used

$$\begin{aligned} \sum_{m=1}^l h \int_{\Omega} (u_{\alpha}^m)^2 dx &= \sum_{m=1}^l h \frac{1}{h^2} \int_{\Omega} \left(\int_{(m-1)h}^{mh} \alpha(x, t) u(x, t) dt \right)^2 dx \\ &\stackrel{\text{H\"older}}{\leq} \sum_{m=1}^l \int_{(m-1)h}^{mh} \int_{\Omega} \alpha^2 u^2 dx dt \leq \bar{\alpha}^2 \int_0^T \int_{\Omega} u^2 dx dt. \end{aligned}$$

Next, we apply the identity $|a|^2 - a \cdot b = \frac{1}{2} |a - b|^2 + \frac{1}{2} |a|^2 - \frac{1}{2} |b|^2$ to I_2 ,

$$\begin{aligned} \sum_{m=1}^l \int_{\Omega} \nabla y^m \nabla (y^m - y^{m-1}) dx &= \sum_{m=1}^l \left(\int_{\Omega} |\nabla y^m|^2 dx - \int_{\Omega} \nabla y^m \nabla y^{m-1} \right) \\ &= \frac{1}{2} \sum_{m=1}^l \int_{\Omega} |\nabla y^m - y^{m-1}|^2 dx + \frac{1}{2} \sum_{m=1}^l \int_{\Omega} (|\nabla y^m|^2 - |\nabla y^{m-1}|^2) dx \\ &= \frac{1}{2} \sum_{m=1}^l \int_{\Omega} |\nabla y^m - \nabla y^{m-1}|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla y^l|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla y^0|^2 dx. \end{aligned}$$

All in all, we obtain

$$\begin{aligned} & \frac{1}{2} \sum_{m=1}^M h \|\delta_h y^m\|_{L^2(\Omega)}^2 + \frac{k}{2} \max_{1 \leq l \leq M} \|\nabla y^m\|_{L^2(\Omega)}^2 \\ & + \frac{k}{2} \sum_{m=1}^l \int_{\Omega} |\nabla y^m - \nabla y^{m-1}|^2 dx \leq \frac{\bar{\alpha}^2}{2} \int_0^T \int_{\Omega} u^2 dx dt + \frac{k}{2} \int_{\Omega} |\nabla y_0|^2 dx . \end{aligned} \tag{10}$$

Define linear and constant in time interpolating functions

$$y_h(t) = y^m + \frac{t - mh}{h} (y^m - y^{m-1}) \quad \text{for } t \in [(m-1)h, mh]$$

$$\hat{y}_h(t) = y^m, \text{ for } t \in [(m-1)h, mh]$$

From (10) we get

$$\|y_{h,t}\|_{L^2(0,T; L^2(\Omega))} + \|\nabla \hat{y}_h\|_{L^2(\Omega)} \leq C_1.$$

Moreover, as for the time-continuous case, see (7), we obtain

$$\begin{aligned} \|y_h(t)\|_{L^2(\Omega)}^2 &\leq 2 \int_{\Omega} y_0^2 dx + 2T \sum_{m=1}^M \int_{(m-1)h}^{mh} \int_{\Omega} (y_{h,t})^2 dx dt \\ &\leq 2 \int_{\Omega} y_0^2 dx + 2T \sum_{m=1}^M h \int_{\Omega} (\delta_h y^m)^2 dx \leq C_2. \end{aligned}$$

Using triangle inequality, we get $\|\nabla y_h(t)\|_{L^2(\Omega)} \leq C_3$,
altogether, we have

$$\|y_h\|_{H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega))} + \|\hat{y}_h\|_{L^\infty(0,T; H^1(\Omega))} \leq C_4. \quad (11)$$

From (10) we infer

$$\begin{aligned} \|\nabla y_h - \nabla \hat{y}_h\|_{L^2(0,T; L^2(\Omega))}^2 &= \int_0^T \int_{\Omega} |\nabla y_h - \nabla \hat{y}_h|^2 dx dt \\ &= \sum_{m=1}^M \int_{(m-1)h}^{mh} \int_{\Omega} \left(\nabla y^m + \frac{t-mh}{h} (\nabla y^m - \nabla y^{m-1}) - \nabla y^m \right)^2 dx dt \\ &= \sum_{m=1}^M \int_{(m-1)h}^{mh} \left(\frac{t-mh}{h} \right)^2 dt \int_{\Omega} |\nabla y^m - \nabla y^{m-1}|^2 dx \\ &= \frac{h}{3} \sum_{m=1}^M \int_{\Omega} |\nabla y^m - \nabla y^{m-1}|^2 dx \xrightarrow{h \rightarrow 0} 0. \end{aligned} \quad (12)$$

Utilizing (11) we get

$$\begin{aligned}
 y_h &\longrightarrow y \quad \text{schwach}^* \text{ in } L^\infty(0, T; H^1(\Omega)) \\
 &\quad \text{schwach in } H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))
 \end{aligned}
 \tag{13}$$

$$\begin{aligned}
 \hat{y}_h &\longrightarrow \hat{y} \quad \text{schwach}^* \text{ in } L^\infty(0, T; H^1(\Omega)) \\
 &\quad \text{schwach in } L^2(0, T; H^1(\Omega)).
 \end{aligned}
 \tag{14}$$

Moreover, we observe

$$\begin{aligned}
 \int_0^T \int_\Omega (y_h - \hat{y}_h)^2 dx dt &= \sum_{m=1}^M \int_{(m-1)h}^{mh} \int_\Omega \frac{(t - mh)^2}{h^2} (y^m - y^{m-1})^2 dx dt \\
 &= \frac{h^2}{3} \sum_{m=1}^M h \int_\Omega (\delta_h y^m)^2 dx \longrightarrow 0.
 \end{aligned}$$

and together with (12)

$$\|y_h - \hat{y}_h\|_{L^2(0, T; H^1(\Omega))}^2 \xrightarrow{h \rightarrow 0} 0, \quad \text{i.e., } y = \hat{y} \quad \text{a.e. in } Q.$$

Now, we get back to (4). Multiplying with $\psi \in L^2(0, T)$ with test function $\phi = \psi\varphi \in L^2(Q)$ we get

$$\int_0^T \int_{\Omega} y_{h,t} \phi \, dx dt + k \int_0^T \int_{\Omega} \nabla \hat{y}_h \nabla \phi \, dx dt = \int_0^T \int_{\Omega} \hat{u}_{\alpha}^h \phi \, dx dt$$

where $\hat{u}_{\alpha}^h(t) = u_{\alpha}^m(x)$ for $t \in ((m-1)h, mh]$.

There holds

$$\hat{u}_{\alpha}^h \rightarrow \alpha u \quad \text{strongly in } L^2(Q).$$

Utilizing (13), (14) and $y = \hat{y}$ a.e. in Q we obtain

$$\int_0^T \left(\int_{\Omega} y_t \varphi dx + k \int_{\Omega} \nabla y \nabla \varphi dx - \int_{\Omega} \alpha u \phi dx \right) \psi(t) dt = 0 \quad \text{for all } \psi \in L^2(0, T)$$

Thus, y solves (4)–(5).