# Optimization II

Lecture Tuesday, April 18, 2023

# Lecture plan

now: Introduction to parabolic Opt. Cont.

today 6-8pm: Analysis for parab. eqns.

Friday, April 21: Numerics for parab. eqns.

Monday " 24: Numerical opt. contr. for "

Thursday " 27: (optional)

Friday " 28: " "

Tuesday, May 2nd: Applications

---

## Chapter 5. Parabolic Optimal Control

### 1. Parabolic state equations

- assume $\Omega \subset \mathbb{R}^n$ with smooth boundary consider parabolic equation

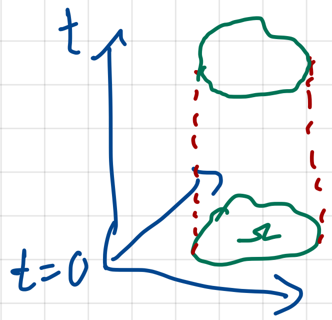
$$y_t - \operatorname{div}(k(x)\operatorname{grad} y) = \alpha(x)\, u, \quad \text{in } \Omega \times (0,T)$$

$$-k\frac{\partial y}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0,T)$$

$$y(0) = y_0, \quad \text{in } \Omega$$

$T$: end time

$Q := \Omega \times (0,T]$ space-time cylinder




**Definition:** Let $I \subset \mathbb{R}$ be an interval and $H$ a Hilbert space. then $C(I,H)$ is the set of all continuous functions $f: I \to H$, $t \mapsto f(t) \in H$, with norm $\|u\| = \sup_{t \in I} \|u(t)\|_H$

Remarks:

(1) $f \in C(I, H)$ is not necessarily continuous in space, e.g., $g \in C[0,T]$, $u \in L^2(\Omega)$, then $f(x,t) = g(t)\, u(x)$ not cont. in $x$

(2) Introduce $L^p(I, H)$ as a completion of $C(I, H)$ with respect to the norm

$$\|f\|_{L^p(0,T;H)} = \left( \int_0^T \|f(t)\|_H^p \right)^{1/p}$$

Examples:

- $C([0,T]; L^2(\Omega))$ with norm

$$\|f\| = \sup_{t \in [0,T]} \left( \int_\Omega f^2(x,t)\, dx \right)^{1/2}$$

- $H^1((0,T); L^2(\Omega))$ with norm

$$\|f\| = \left( \int_0^T \left( \|f\|^2_{L^2(\Omega)} + \|f_t\|^2_{L^2(\Omega)} \right) dt \right)^{1/2}$$

__Weak formulation__ , assume $k$ constant

$$(SE) \begin{cases} \int_\Omega y_t(t)\, \varphi \, dx + \overbrace{k \int_\Omega \nabla y(t) \nabla \varphi \, dx}^{a(y,\varphi)} = \int_\Omega \alpha u \, \varphi \, dx \\ \qquad \forall \varphi \in H^1(\Omega) \text{ and } t \in (0,T) \text{ a.e.} \\ y(0) = y_0 \text{ in } \Omega \end{cases}$$

We will show, (SE) has a unique sol.

$$\bar{y} \in H^{1,1}(Q) = H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

$$= \{ f \in L^2(Q),\ u_{x_i} \in L^2(\Omega),\ 1 \le i \le n,\ u_t \in L^2(\Omega) \}$$

- two strategies to prove existence of solutions
  (1) Galerkin approach ( Evans's Book )
  (2) implicite time discretization ( Rothe method)
  idea: replace $Y_t(t)$ by difference quotient $\dfrac{Y(t+h)-Y(t)}{h}$

2. The parabolic optimal control problem

$$J(y,u) = \frac{1}{2}\int_0^T\int_\Omega (y-y_d^Q)^2\,dx\,dt + \frac{\gamma_1}{2}\int_\Omega (y(T)-y_d^\Omega)^2\,dx + \frac{\gamma_2}{2}\int_0^T\int_\Omega u^2\,dx\,dt$$

then the control problem is

$$(CP)\quad \begin{cases} \min J(y,u) \\ \quad \text{subj. to } y \text{ sol to (SE)} \\ \qquad \text{and } u \in \mathcal{U}_{ad} \subset L^2(Q) \end{cases}$$

Remark: Under standard assumptions, existence of an opt. control can be proven as in the elliptic case.

Now, derive optimality system using the Lagrangean:

$$\mathcal{L}(y,p,u) = J(y,u) - \int_0^T\int_\Omega y_t\, p\, dx\,dt - k\int_0^T\int_\Omega \nabla y\cdot\nabla p\, dx\,dt + \int_0^T\int_\Omega \alpha u p$$

Now take direction $h$ such that $h(x,0)=0$ and compute $\mathcal{L}_y(\bar{y},p,\bar{u})h = 0$ to get adjoint eqn.

$$0 = \int_0^T\int_\Omega \cancel{(\bar{y}-y_d^Q)}\, h\, dx\,dt + \gamma_1\int_\Omega (\bar{y}(T)-y_d^\Omega)h(T)\,dx \underbrace{- \int_0^T\int_\Omega h_t\, p\, dx\,dt}_{= \int_0^T\int_\Omega \cancel{h p_t}\, dx\,dt - \int_\Omega h(t)p\Big|_0^T} \underbrace{- k\int_0^T\int_\Omega \nabla h\cdot\nabla p\, dx\,dt}_{k\int_0^T\int_\Omega \cancel{h\Delta p}\, dx\,dt - k\int_0^T\underbrace{\int_{\partial\Omega}}_{=0}\frac{\partial p}{\partial\nu}h\, dx\,dt}$$

this leads to

$$\int_0^T \int_\Omega h \underbrace{\left[ y - y_d^Q + p_t + k\Delta p \right]}_{=0} dx$$

$$+ \int_\Omega h(T) \underbrace{\left( -p(T) + \gamma_1 (\bar{y}(T) - y_d^\Omega) \right)}_{=0} dx = 0$$

Strong form of adj. equation

$$(AE) \begin{cases} -p_t - k\Delta p = \bar{y} - y_d^Q & \text{in } \Omega \times (0,T) \\ -k \frac{\partial p}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ p(T) = \gamma_1 (\bar{y}(T) - y_d^\Omega) \end{cases}$$

Remark: Adjoint eqn is backward in time well-posed parabolic eqn.. By substituting $\tilde{p}(t,x) = p(T-t,x)$ we get a usual probl..

To compute the gradient, we again use

the Lagrangian to obtain.

$$\mathcal{L}_u(\bar{y}, p, \bar{u})h = \int_0^T \int_\Omega (\gamma_2 \bar{u} + \alpha p) h \, dx \, dt$$

Using the Riesz algorithm, we get

$$f'(u) = \gamma_2 \bar{u} + \alpha p$$

and the variational inequ.

$$\int_0^T \int_\Omega (\gamma_2 \bar{u} + \alpha p)(u - \bar{u}) \, dx \, dt \geq 0$$

$$\forall u \in U_{ad} \subset L^2(0,T, L^2(\Omega))$$

convex and closed.

Remark:

For box constraints, we get a projection formula proceeding exactly as in the elliptic case.