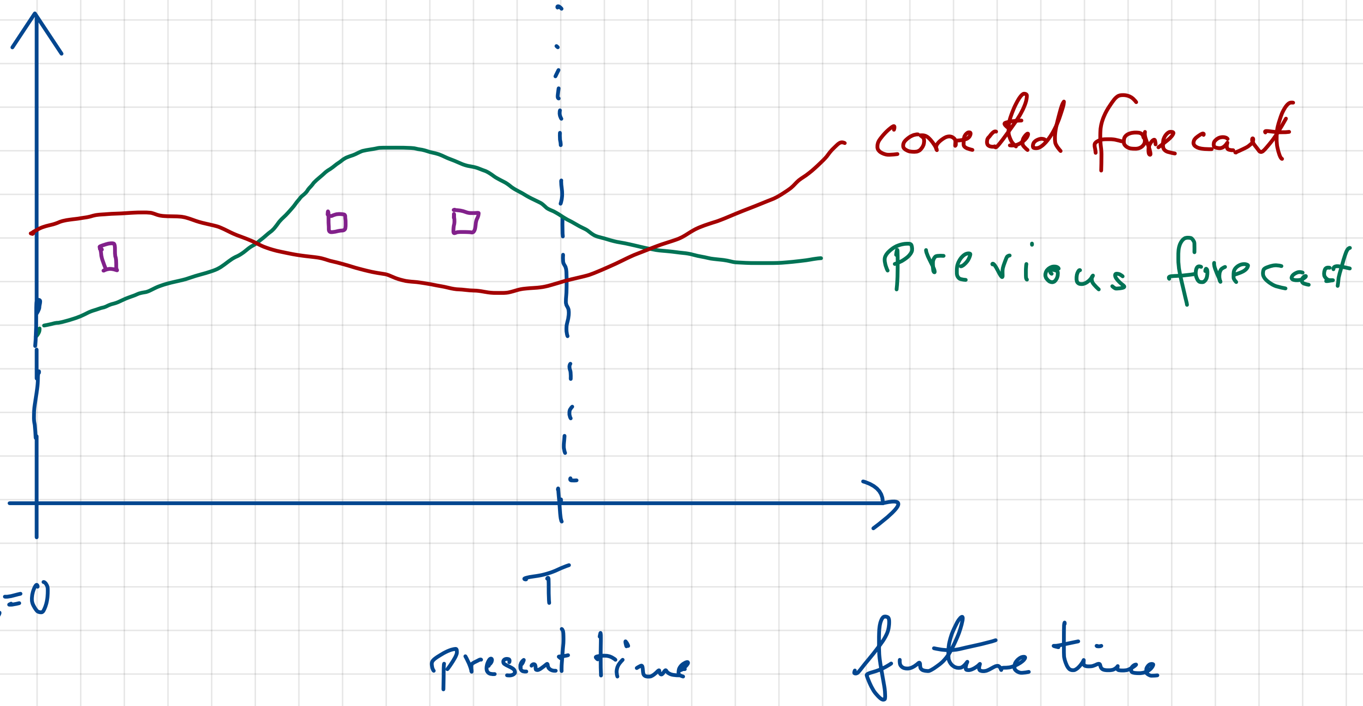
Optimization II

Lecture Tuesday, May 2, 2023

Variational data assimilation

Goal: calibrate your model to improve predictions ⇝ forecasting



two stages:

Step 1: analysis step
observations of previous and present time assimilate to derive optimal trajectory

Step 2: forecasting
Hope: better fit of past data leads to better predictions

$\rightsquigarrow$ iterate these steps.

Abstract formulation:

(SE) $\quad y_t + A(u; y) = L(u) \quad$ in $\Omega \times (0,T)$

$$y(0) = y_0 \quad \text{in } \Omega$$

control $\quad c(x,t) = (y_0(x), u(x,t))$

cost functional:

$$J(c,y) = \frac{1}{2} \int_0^T \| C(y(t)) - z^{ob} \|_R^2 \, dt$$

$$+ \frac{\mu_0}{2} \| y_0 - y_0^d \|_H^2 + \frac{\mu_1}{2} J_{reg}(u)$$

C: observation operator

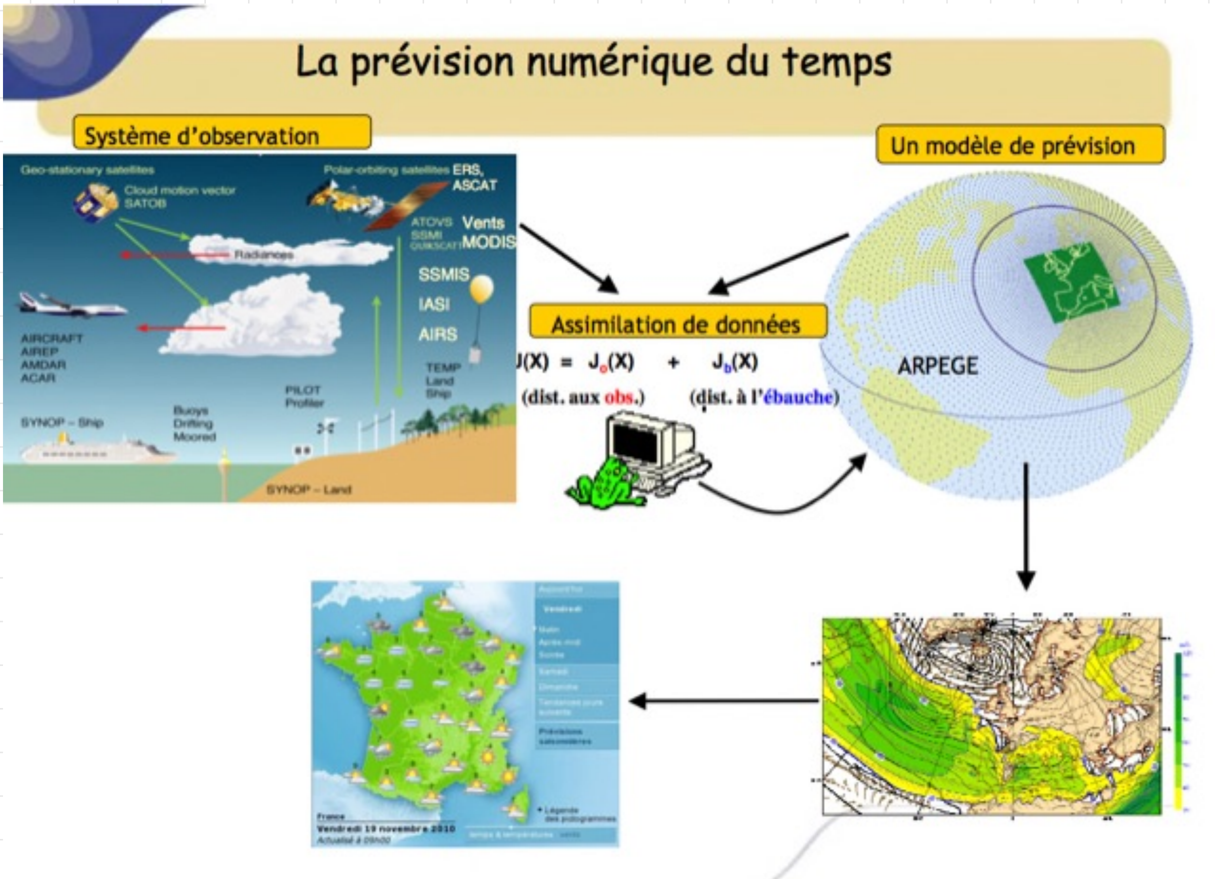
$J_{reg.}$ regularization term

data assim. problem:

Find $\tilde{c}(x,t) = (\tilde{y}_0(x), \tilde{u}(x,t))$ s.t.

$$j(\tilde{c}) = \min_{c \in \mathcal{C}_{ad}} \bar{j}(c)$$

Example (Weather forecast)



- Very challenging.
- adjoint based methods since 80s
- since 2000s VDA implemented in all large weather pred. centers
- model based on conservation laws

(energy, mass, momentum)

- unknowns: temperature, humidity, pressure, velocities, ...

- each time step is about 10 mins for mid range forecast

Ref. Jerome Monnier: Var. Data Assimilation and Model learning. France 2021.

Remark: A further very important and timely application of VDA are digital twins.

Let's consider a simple case:

$$\min J(y,\omega) = \frac{1}{2}\int_0^T\int_\Omega (y - y_d^Q)^2 \, dx \, dt + \frac{\gamma}{2}\int_\Omega (\omega - y_0^d)^2 \, dx$$

subject to

$$y_t - \Delta y = f \quad \text{in } \Omega \times (0,T)$$

$$-\frac{\partial y}{\partial \nu} = y \quad \text{in } \partial\Omega \times (0,T)$$

$$y(0) = w \quad \text{in } \Omega$$

and $w \in \mathcal{U}_{ad} \subset L^2(\Omega)$

Remark: One can show that for any $f \in L^2(Q)$ and $w \in L^2(\Omega)$ there exists a unique sol. y of (SE), i.e., solution operator $w \longmapsto y(w)$ is well-defined.

Goal: derivation of the optimality system.

Step 1 Derive weak form for (SE)

$$\int_0^T\int_\Omega (y_t - \Delta y - f)\,\varphi\,dx$$

$$= -\int_0^T\int_\Omega y\,\varphi_t\,dx\,dt + \int_\Omega y(T)\,\varphi(T)\,dx$$

$$- \int_\Omega \omega\,\varphi(0)\,dx + \int_0^T\int_\Omega \nabla y\,\nabla\varphi\,dx$$

$$+ \int_0^T\int_{\partial\Omega} y\,\varphi\,dx\,ds - \int_0^T\int_\Omega f\,\varphi\,dx\,dt = 0$$

Lagrangian:

$$\mathcal{L}(y,p,\omega) = \frac{1}{2}\int_0^T\int_\Omega (y - y_d^Q)^2\,dx\,dt + \frac{\gamma}{2}\int_\Omega (\omega - y_0^d)^2\,dx$$

$$+ \int_0^T\int_\Omega y\,p_t\,dx\,dt - \int_\Omega y(T)\,p(T)\,dx$$

$$+ \int_\Omega \omega\,p(0)\,dx - \int_0^T\int_\Omega \nabla y\,\nabla p\,dx$$

$$- \int_0^T\int_{\partial\Omega} y\,p\,dx\,ds + \int_0^T\int_\Omega f\,p\,dx\,dt$$

Step 2:
for adj. equ. compute $\mathcal{L}_Y h = 0 \rightsquigarrow$

$$0 = \int_0^T \int_\Omega (Y - Y_d) h \, dx\, dt + \int_0^T \int_\Omega h p_t \, dx\, dt$$

$$- \int_\Omega h(T) p(T) \, dx - \underbrace{\int_0^T \int_\Omega \nabla h \cdot \nabla p \, dx\, dt}_{\int_0^T \int_\Omega h \Delta p \, dx\, dt - \int_0^T \int_{\partial\Omega} h \frac{\partial p}{\partial \nu} dx\, dt} - \int_0^T \int_{\partial\Omega} h p \, dx\, dt$$

$$\Rightarrow \int_0^T \int_\Omega h \, [\underbrace{Y - Y_d + p_t + \Delta p}_{=0}] \, dx\, dt + \int_0^T \int_{\partial\Omega} h \, [\underbrace{-p - \frac{\partial p}{\partial \nu}}_{=0}] \, dx\, dt$$

$$- \int_\Omega h(T) \underbrace{p(T)}_{=0} \, dx = 0$$

(AE) $\quad -p_t - \Delta p = Y - Y_d \quad$ in $\Omega \times (0, T)$

$\qquad -\frac{\partial p}{\partial \nu} = p \quad$ in $\partial\Omega \times (0, T)$

$\qquad p(T) = 0$

Step 3: Compute gradient and (VI)

$$j'(w)h = \mathcal{L}_w(\bar{y}, p, \bar{w})h$$

$$= \int_\Omega \gamma(\bar{w} - Y_0^d) h \, dx + \int_\Omega h \, \bar{p}(0) \, dx$$

$$= \int_\Omega h \left[\gamma(\bar{w} - Y_0^d) + \bar{p}(0) \right] dx$$

apply Riesz to get L^2 gradient

$$j'(\bar{w}) = \gamma(\bar{w} - Y_0^d) + \bar{p}(0)$$

and variational inequality

$$\int_\Omega \left(\gamma(\bar{w} - Y_0^d) + \bar{p}(0) \right)(w - \bar{w}) \, dx \geq 0$$

for all $w \in W_{ad}$

Remarks about oral exam:

May 24th, 30-45 mins

- well posedness of state eqn (elliptic)
- existence of optimal controls (u)
- Necess. opt. cond. (ell. + parab.)
- Numerics (ell. + parab.)

Hint: Have a look at derivation of opt. cond. for

- project
- Worked out example for numerics (lect. March 16, section 6.1 Test Example