Optimization II

Lecture Thursday, March 16

6.1 A test example

$$\min J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \int_\Gamma e_\Gamma y \, ds + \frac{1}{2}\int_\Omega u^2 dx$$

$$\text{s.t. } (SE) \begin{cases} -\Delta y + y = u + e_\Omega & \text{in } \Omega \\ \frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma := \partial\Omega \end{cases}$$

and $0 \le u(x) \le 1$ a.e. in Ω

Weak form. of (SE)

$$\int_\Omega \nabla y \nabla \varphi \, dx + \int_\Omega y \varphi \, dx = \int_\Omega (u + e_\Omega) \varphi \, dx$$

$$\forall \varphi \in H^1(\Omega)$$

Lagrangian:

$$\mathcal{L}(y,p,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \int_\Gamma e_\Gamma y dx + \frac{1}{2}\int_\Omega u^2 dx$$

$$- \int_\Omega \nabla y \nabla p dx - \int_\Omega y p dx + \int_\Omega (u + e_\Omega) p dx$$

AE, compute $\mathcal{L}_y(\bar{y}, p, \bar{u}) h = 0$:

$$\int_\Omega (\bar{y} - y_d) h dx + \int_\Gamma e_\Gamma \bar{y} dx - \underbrace{\int_\Omega \nabla h \nabla p dx - \int_\Omega h p dx}_{\int_\Omega \Delta p h dx - \int_{\partial\Omega} \frac{\partial p}{\partial \nu} h dx} = 0$$

$\Rightarrow$

$$(AE) \begin{cases} -\Delta p + p = \bar{y} - y_d & \text{in } \Omega \\ \frac{\partial p}{\partial \nu} = e_\Gamma & \text{on } \partial\Omega \end{cases}$$

<u>Theorem 23</u> $\Longrightarrow$ $\bar{u} = \mathbb{P}_{[0,1]}(-p(x))$ a.e. in Ω

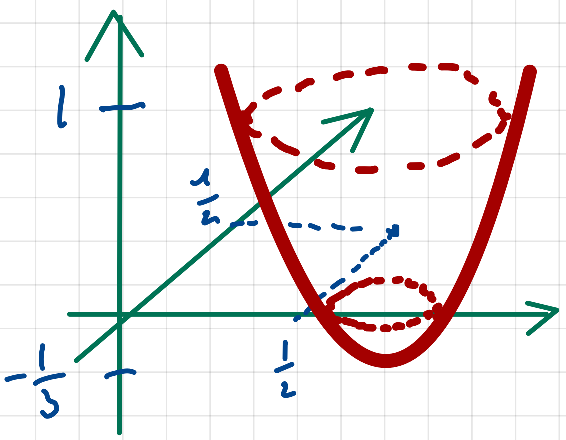
Goal: compute sol. $\bar{u}$ as cut-off rotational paraboloid with vertex $\hat{x} = (+\frac{1}{2}, +\frac{1}{2})^T$

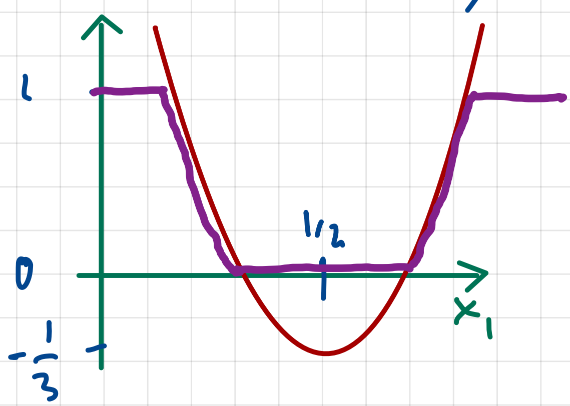
define $\Omega = (0,1) \times (0,1)$,

and $r(x) = |x - \hat{x}| = \sqrt{(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2}$

and then

$$\bar{u} = P_{[0,1]} \left(12 r^2 - \frac{1}{3} \right)$$





hence

$$p = -12 r^2 + \frac{1}{3} = -12 \left(x_1 - \frac{1}{2} \right)^2 - 12 \left(x_2 - \frac{1}{2} \right)^2 + \frac{1}{3}$$

choose $\bar{y} \equiv 1 \Rightarrow \frac{\partial \bar{y}}{\partial \nu} = 0$ on $\partial \Omega$

and $-\Delta \bar{y} + \bar{y} = 1 \overset{!}{=} \bar{u} + e_\Omega$

$$\Rightarrow e_\Omega = 1 - \bar{u}$$

Moreover, we obtain

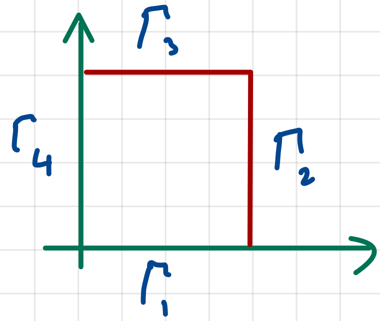
$$-\Delta p + p = 48 - 12r^2 + \tfrac{1}{3} \overset{!}{=} \bar{y} - y_d = 1 - y_d$$

$$\Rightarrow y_d = -47 - \tfrac{1}{3} + 12r^2 = -\tfrac{142}{3} + 12r^2$$

boundary condition for p:

we have

$$\frac{\partial p}{\partial \nu} = \nabla p \cdot \nu = -24 \begin{pmatrix} x_1 - \frac{1}{2} \\ x_2 - \frac{1}{2} \end{pmatrix} \cdot \nu \overset{!}{=} e_\Gamma$$



on Γ_1 : $\nu = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $x_2 = 0$

$$\Rightarrow e_\Gamma = -12$$

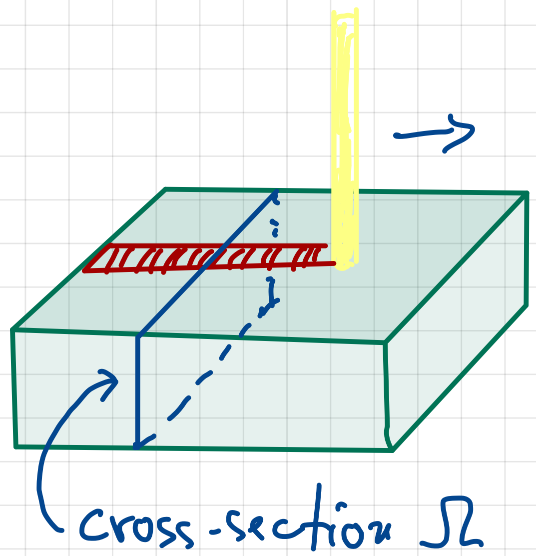
on Γ_2 : $\nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_1 = 1 \Rightarrow e_\Gamma = -12$

on Γ_3 : $\nu = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_2 = 1 \Rightarrow e_\Gamma = -12$

on Γ_4 : $\nu = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $x_1 = 0 \Rightarrow e_\Gamma = -12$

6.2 Laser beam shaping

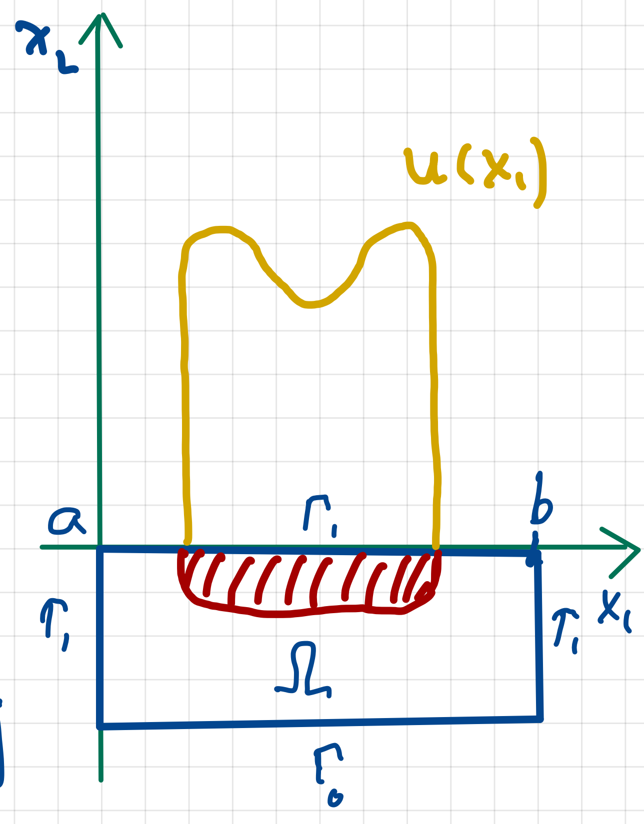
(see additional slides)



- laser moves along workpiece surface
- causes heating pattern
- Goal: achieve desired heating pattern

Simplification:

- long workpiece
- stationary heat cond. in slowly moving cross section $\Omega \subset \mathbb{R}^2$
- Goal: find intensity $u(x_1)$, such that heating pattern is rectangular



- Math. model (stationary heat equation)

$$(SE)\quad \begin{cases} -\operatorname{div}(k(x)\operatorname{grad} y) = f & \text{in } \Omega \\ -k\dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ y = y_0 & \text{on } \Gamma_0 \end{cases}$$

where y temperature

k heat conductivity

y_0 substrate temperature

f distributed heat source

$$f(x_1, x_2) = u(x_1)\, e^{\alpha x_2} \quad \text{with } \alpha > 0 \text{ and } x_2 \le 0$$

u : control, intensity of laser beam

- Cost functional

$$J(y,u) = \frac{1}{2}\int_{\Omega} (y-y_d)^2\,dx + \frac{\gamma}{2}\int_a^b u(x_1)^2\,dx_1$$

(CP) $\min J(y,u)$

s.th. y solve (SE)

and $0 \le u(x_1) \le u_{max}$

a.e. in (a,b)