Optimization II

Lecture 9, Feb. 2

Revision

(CPI) $\quad \min \underbrace{\frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\gamma}{2}\int_\Omega u^2\,dx}_{J(y,u}$

s.t.

(SE) $\displaystyle\int_\Omega \nabla y\cdot\nabla\varphi\,dx = \int_\Omega u\varphi\,dx \quad \forall \varphi\in H_0^1(\Omega)$

$u \in \mathcal{U}_{ad} \subset L^2(\Omega)$

(closed, convex, bounded)

- (LM) $\Longrightarrow$ for any $u\in\mathcal{U}_{ad}$, (SE) has a unique sol. $y(u)$

- define control-to-state operator
  $G: u \longmapsto y(u)$, $\quad L^2(\Omega)\to H_0^1(\Omega)$

- introduce sol. operator

$$S = E_2 \circ G \;, \quad E_2 : H^1(\Omega) \to L^2(\Omega)$$

(cont. embedding)

$$S : u \mapsto y \;, \quad L^2(\Omega) \to L^2(\Omega)$$

- define reduced cost functional

$$f(u) = J(S(u), u)$$

$$f : \mathcal{U}_{ad} \subset L^2(\Omega) \to \mathbb{R}$$

then $(CP1) \iff \min\limits_{u \in \mathcal{U}_{ad}} f(u) \quad (*)$

- $(CP1)$ has a unique sol. $\bar{u} \in \mathcal{U}_{ad}$
- let $\bar{u}$ be sol to $(*)$ and assume f is F-diffb., then

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall\, u \in \mathcal{U}_{ad}$$

where $f' : L^2(\Omega) \to L(L^2(\Omega), \mathbb{R})$

- for $f(u) = \frac{1}{2}\int_\Omega (S(u) - y_d)^2 dx + \frac{\gamma}{2}\int_\Omega u^2 dx$

$$= \frac{1}{2}\| S(u) - y_d \|_{L^2}^2 + \frac{\gamma}{2}\|u\|_{L^2}^2$$

$$f'(u)h = (S(u) - y_d, Sh)_{L^2} + \gamma (u, h)_{L^2}$$

$$= (\underbrace{S^*(S(u) - y_d)}_{p}, h)_{L^2} + \gamma (u, h)_{L^2}$$

$$= (p + \gamma u, h)_{L^2}$$

where $p = S^*(y - y_d)$ is the sol to

$$\begin{cases} (AE) & -\Delta p = y - y_d \quad \text{in } \Omega \\ & p = 0 \quad \text{on } \partial\Omega \end{cases}$$

Theorem 18: Let $\gamma \geq 0$, $\mathcal{U}_{ad} \subset L^2(\Omega)$

closed and convex and either bounded or $\gamma > 0$, $y_d \in L^2(\Omega)$, then $\bar{u} \in \mathcal{U}_{ad}$ is a sol to (CP1) with optimal state $\bar{y}$ if and only if

(1) $$a_0(\underbrace{y(\bar{u})}_{=\bar{y}}, \varphi) = \int_\Omega u\varphi \quad \forall \varphi \in H_0^1(\Omega)$$

(2) there exists an adjoint $\bar{p} \in H_0^1(\Omega)$ as unique sol to

(AE) $$a_0(\bar{p}, \varphi) = \int (\bar{y} - y_d)\varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$$

(3) $$(\bar{p} + \gamma \bar{u}, u - \bar{u})_{L^2} \geq 0 \quad \forall u \in \mathcal{U}_{ad}$$

(variational inequality)

If $\mathcal{U}_{ad} = L^2(\Omega)$, then take

$$u - \bar{u} = -(\bar{p} + \gamma \bar{u})$$

$$\Rightarrow -\|\bar{p}+\gamma\bar{u}\|^2 \geq 0$$

$$\Rightarrow \quad \bar{u} = -\frac{1}{\gamma}\bar{p}$$

Corollary 19: Let $U_{ad} = L^2(\Omega)$ and $\gamma > 0$, then $\bar{u}$ is the sol to $((CP1)) \Leftrightarrow$

$(\bar{y},\bar{p}) \in H_0^1(\Omega)\times H_0^1(\Omega)$ solves

$$a_0(\bar{y},\varphi) = -\frac{1}{\gamma}\int_\Omega \bar{p}\varphi\,dx \quad \forall\varphi\in H_0^1(\Omega)$$

$$a_0(\bar{p},\varphi) = \int_\Omega (y-y_d)\varphi\,dx \quad \forall\varphi\in H_0^1(\Omega)$$

<u>Remark:</u> An easy way of deriving the optimality system is by using the Lagrangian, i.e.,

$$\mathcal{L}(y,p,u) = J(y,u) - a_0(y,p) + (u,p)$$

$$= \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\gamma}{2}\int_\Omega u^2\,dx - \int_\Omega \nabla y\cdot\nabla p\,dx + \int_\Omega up\,dx$$

As in finite dimensions (cf. lect. 2), we will see that

(1) $\mathcal{L}_y(\bar{y}, \bar{p}, \bar{u})h = 0 \rightsquigarrow$ (AE)

(2) $\mathcal{L}_p(\bar{y}, \bar{p}, \bar{u})h = 0 \rightsquigarrow$ (SE)

(3) $\mathcal{L}_u(\bar{y}, \bar{p}, \bar{u})(u - \bar{u}) \geq 0 \rightsquigarrow$ (VI)

indeed,

$$\mathcal{L}(y, p, u) = \frac{1}{2}\int_\Omega (y - y_d)^2 dx + \frac{\gamma}{2}\int_\Omega u^2 dx - \int_\Omega \nabla y \cdot \nabla p \, dx + \int_\Omega u p \, dx$$

$$\mathcal{L}_p(\bar{y}, \bar{p}, \bar{u})h \overset{!}{=} 0 \implies$$

$$0 = -\int_\Omega \nabla y \cdot \nabla h \, dx + \int_\Omega u h \, dx \quad \forall h \in H_0^1(\Omega)$$

(SE)

$$\mathcal{L}_y(\bar{y}, \bar{p}, \bar{u})h \overset{!}{=} 0$$

$$0 = \int_\Omega (y - y_d) h \, dx - \int_\Omega \nabla h \cdot \nabla p \, dx \quad \forall h \in H_0^1(\Omega)$$

$$\mathcal{L}(y,p,u) = \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\gamma}{2}\int_\Omega u^2\,dx - \int_\Omega \nabla y\cdot\nabla p\,dx + \int_\Omega u p\,dx$$

$$\mathcal{L}_u(\bar{y},\bar{p},\bar{u})h = f'(\bar{u})h$$

$$= \gamma\int_\Omega \bar{u}h\,dx + \int_\Omega h\bar{p}\,dx = \int_\Omega (\gamma\bar{u}+\bar{p})h\,dx$$

Now, we try to further exploit (VI)

$$(\text{VI}) \quad (\bar{p}+\gamma\bar{u},\, u-\bar{u})_{L^2} \geq 0$$

$$\iff (\bar{p}+\gamma\bar{u},\, \bar{u})_{L^2} \leq (\bar{p}+\gamma\bar{u},\, u)_{L^2}$$

hence $(\bar{p}+\gamma\bar{u},\bar{u}) = \min\limits_{u\in\mathcal{U}_{ad}} (\bar{p}+\gamma\bar{u},u)_{L^2}$

Next, we introduce box constraints:

$$\mathcal{U}_{ad}^B = \{u\in L^2(\Omega) \mid \xi_0(x)\leq u(x)\leq \xi_1(x), \text{a.e. in } \Omega\}$$

with $\xi_0, \xi_1 \in L^\infty$

then $\mathcal{U}_{ad}^{B}$ is closed, convex and bounded in $L^2(\Omega)$

Now, we show, that (VI) can be discussed pointwise a.e. in Ω:

<u>Lemma 20.</u> (VI) $\Longleftrightarrow$

$$(\bar{p}(x) + \gamma \bar{u}(x))(\xi - \bar{u}(x)) \geq 0$$

$$\forall \xi \in [\xi_0(x), \xi_1(x)] \quad \text{a.e. in } \Omega$$

<u>Proof:</u> Define $z(x) = \bar{p}(x) + \gamma u(\bar{x})$,

z is Lebesgue measurable, hence almost all $x_0 \in \Omega$ are Lebesgue points i.e., there holds

$$\lim_{\rho \searrow 0} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} z(x)\,dx = z(x_0)$$

For ρ small enough, $B_\rho(x_0) \subset \Omega$

now choose
$$u(x) = \begin{cases} \xi, & \text{in } B_\rho(x_0) \subset \Omega \\ \bar{u}, & \text{else} \end{cases}$$

where $\xi \in [\varphi_0(x_0), \varphi_1(x_0)]$

then $u \in \mathcal{U}_{ad}^{B}$ and

$$0 \leq \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} z(x) \, (\underbrace{u}_{=\xi} - \bar{u}(x)) \, dx$$

$$\Downarrow$$

$$z(x_0)(\xi - \bar{u}(x_0))$$

$\square$

<u>Corollary:</u> $\bar{u} \in \mathcal{U}_{ad}^{B}$ be optimal

$\Longleftrightarrow$ (1) or (2) is satisfied

$$(1) \quad \min_{\xi \in [\varphi_0(x), \varphi_1(x)]} (\bar{p}(x) \cdot \gamma \bar{u}(x)) \xi$$

$$= (\bar{p}(x) + \gamma \bar{u}(x)) \, \bar{u}(x)$$

(2) $$\min_{\xi \in [\xi_0(x), \xi_1(x)]} \left(p(x)\xi + \frac{\gamma}{2}\xi^2 \right)$$

$$= p(x)\,\bar{u}(x) + \frac{\gamma}{2}\,\bar{u}(x)^2$$