Optimization II

Lecture 7, Jan. 27

2. Weak convergence in Hilbert spaces

Def. 4 A sequence $\{u_n\}$ in a Hilbert space H is said to converge weakly to $u \in H$, $u_n \longrightarrow u$

$$\Longleftrightarrow \quad f(u_n) \to f(u) \quad \forall f \in H^*$$

$$\Longleftrightarrow \quad (u_n, v)_H \longrightarrow (u, v) \quad \forall v \in H$$

Lemma 5: Let $\{u_n\} \subset H$, then

(1) $u_n \to u$ (strongly) $\Longrightarrow$ $u_n \longrightarrow u$ (weakly)

(2) $\{u_n\}$ weakly convergent

$\Longrightarrow$ $\{u_n\}$ is bounded

(Banach-Steinhaus Theorem)

Proof of (1)

$u_n \longrightarrow u$ strongly, let $v \in H$

then we Cauchy. Schwartz inequ:

$$0 \leq |(u_n - u, v)_H| \leq \|u_n - u\| \cdot \|v\| \xrightarrow[n \to \infty]{} 0$$

$\square$

Corollary 6: Every weakly sequ. closed set M is also strongly closed.

Proof: Take $\{u_n\} \subset M$, $u_n \longrightarrow u$

$$\xrightarrow{\text{lemma 5}} u_n \longrightarrow u \implies u \in M$$

Example: Consider $H = L^2(0, 2\pi)$

and $M = \{u \in H \mid \|u\| = 1\}$

then M is strongly closed.

Consider sequence $\{u_n\}$

$$u_n(x) = \frac{1}{\sqrt{\pi}} \sin(nx)$$

and compute n-th Fourier coeff. of an $f \in H$, . i.e.,

$$c_n = (f, u_n) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

we have $c_n \xrightarrow[n \to \infty]{} 0$

$$\Rightarrow (f, u_n) \to 0 \quad \text{i.e.} \quad u_n \to 0$$

on the other hand

$$\|u_n\|^2 = \frac{1}{\pi} \int_0^{2\pi} \sin^2(nx)\, dx = 1$$

i.e. $u_n \in \mathcal{U}$

We can conclude:

(1) there are sequences converging weakly to zero but all el. are on the unit sphere

(2) A strongly closed set needs not to

be weakly closed

(3) $\|.\|$ is not weakly sequentially continuous, since $u_n \rightharpoonup 0$ but $\|u_n\| = 1 \neq 0 = \|0\|$

Lemma 7: let $M \subset H$ closed and convex $\Rightarrow$ M is weakly closed

Lemma 8: let $f: H \to \mathbb{R}$ continuous and convex, then f is lower semi-continuous (l.s.c) with respect to weak convergence

$$u_n \rightharpoonup u \Rightarrow \liminf_{n\to\infty} f(u_n) \geq f(u)$$

Example: $f(u) = \|u\|$ convex and continuous.

Lemma 9: Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

3. Unique solvability of (CP1) and (CP2)

Let's start with (CP1):

$$\min \underbrace{\frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\gamma}{2}\int_\Omega u^2\,dx}_{J(y,u)}$$

s.t. $$\underbrace{\int_\Omega \nabla y\cdot\nabla\varphi\,dx}_{a_0(y,\varphi)} = \int_\Omega u\varphi\,dx \quad \forall \varphi\in H_0^1(\Omega)$$

and $u \in \mathcal{U}_{ad} \subset L^2(\Omega)$

Using Lax-Milgram $\Rightarrow$ for any $u \in L^2(\Omega)$ (SE1) has a unique sol. $y = y(u) \in H_0^1(\Omega)$

Hence, we have well-defined control-to-state map

$$G : u \longmapsto y(u) \quad , \quad L^2(\Omega) \to H_0^1(\Omega)$$

Thanks to the embedding

$$E_2 : H^1(\Omega) \longrightarrow L^2(\Omega)$$

we can define

$$S = E_2 \circ G \quad , \quad L^2(\Omega) \to L^2(\Omega)$$

the solution operator

and introduce the reduced cost functional

$$f(u) := J(y(u), u) = \frac{1}{2} \int_\Omega (Su - y_d)^2 dx + \frac{\gamma}{2} \int_\Omega u^2 dx$$

then (CP1) $\Longleftrightarrow$ $\min\limits_{u \in \mathcal{U}_{ad}} f(u)$

i.e. an infinite-dim. optim. probl. in $L^2(\Omega)$

Theorem 10:

Let $\mathcal{U}_{ad} \subset L^2(\Omega)$ closed, convex and bounded, $\gamma \geq 0$ and $y_d \in L^2(\Omega)$.

Then, (CP1) has a unique sol. $\bar{u} \in \mathcal{U}_{ad}$.

Proof: (i) existence of an opt. control

Let $$\ell = \inf_{u \in \mathcal{U}_{ad}} J(y(u), u)$$

and $\{u_n\} \subset \mathcal{U}_{ad}$ minimizing sequence,

i.e. $J(y(u_n), u_n) \longrightarrow \ell$

$\mathcal{U}_{ad}$ bounded $\xRightarrow{\text{Lemma 9}}$ $\exists$ subsequence

$\{u_{n'}\}$, s.th. $u_{n'} \longrightarrow \bar{u}$ weakly

$\mathcal{U}_{ad}$ convex, closed $\xRightarrow{\text{lemma 7}}$ $\bar{u} \in \mathcal{U}_{ad}$

let $y_n = S(u_n)$ sol. to (SE1), i.e.

$$(1) \quad \int_\Omega \nabla y_n \nabla \varphi \, dx = \int_\Omega u_n \varphi \, dx \quad \forall \varphi \in H_0^1$$

Take $\varphi = y_n$ then

$$\|y_n\|^2_{H^1(\Omega)} \leq \int_\Omega |\nabla y_n|^2 dx = \int_\Omega u_n y_n \, dx$$

$$\text{Hölder} \rightarrow \leq \|u_n\|_{L^2} \|y_n\|_{L^2}$$

$$\text{Young} \rightarrow \leq \frac{1}{2}\|u_n\|^2_{L^2} + \frac{1}{2}\|y_n\|$$

$$\Rightarrow \frac{1}{2}\|y_n\|^2_{H^1(\Omega)} \leq \frac{1}{2}\|u_n\|^2_{L^2}$$

$$\xRightarrow{\text{lemma 5 (2)}} \quad \|y_n\|_{H^1(\Omega)} \leq C$$

with some constant C indep. n

$\Rightarrow$ $\exists$ subsequ. $Y_{n''} \longrightarrow \bar{Y}$ in $H^1(\Omega)$

consider (SE1)

$$\begin{tikzcd} \int_\Omega \nabla Y_{n''} \cdot \nabla \varphi \, dx \arrow[d] & = & \int_\Omega u_n \varphi \, dx \arrow[d] \\ \int_\Omega \nabla \bar{Y} \cdot \nabla \varphi \, dx & = & \int_\Omega \bar{u} \varphi \, dx \end{tikzcd}$$

$\Rightarrow$ $Y_{n''} \longrightarrow \bar{Y} = S(\bar{u})$

$f(u) = J(y(u), u)$ is convex

$$\xRightarrow{\text{lemma 8}}$$

$$f(\bar{u}) \leq \liminf_{n \to \infty} f(u_n) \leq \lim_{n \to \infty} J(Y_n, u_n) = \ell$$

$$= \inf_{u \in \mathcal{U}_{ad}} J(y(u), u)$$

$\Rightarrow$ $\bar{u}$ is an opt. control with opt. state $\bar{y}$.

(ii) uniqueness

follows from strict convexity and linearity of sol. operator

[hint:
there holds $\left[\frac{1}{2}(a+b)\right]^2 < \frac{1}{2}a^2 + \frac{1}{2}b^2$

if $a \neq b$]

$\square$

Remark:

(1) The assertion remains correct, if $\mathcal{U}_{ad} = \mathcal{U} = L^2(\Omega)$ and $\gamma > 0$.

Then, there holds

$$J(y,u) = \frac{1}{2}\int (y-y_d)^2\,dx + \frac{\gamma}{2}\int_\Omega u^2\,dx$$

$$\geq \frac{\gamma}{2}\int_\Omega u^2\,dx \qquad (*)$$

take min. sequ, then boundedness follows from (*)

(2) Uniqueness can be shown for any strictly convex $f(u)$

<u>Corollary 11.</u> Let $W_{ad} \subset L^2(\partial\Omega)$ closed, bounded and convex, $y_d \in L^2(\Omega)$, $\gamma \geq 0$, then (CP2) has a unique sol. $\bar{w} \in W_{ad}$ with corresp. opt. state $\bar{y} = S(\bar{w})$.