Optimization II

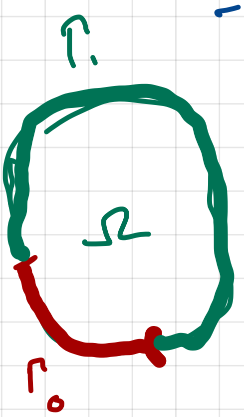
Lecture 6, Jan. 26

Chapter Linear-quadratic elliptic optimal control problems

1. Problem formulation

Let $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, then we consider the elliptic problem

$$
\begin{aligned}
-\Delta y &= u && \text{in } \Omega \\
y &= 0 && \text{on } \Gamma_0 \subset \partial\Omega \\
-\frac{\partial y}{\partial \nu} &= \beta\,(y - w) && \text{on } \Gamma_1 = \partial\Omega \setminus \Gamma_0
\end{aligned}
$$



For the data, we assume

(A1) $\beta \geq 0$ constant

(A2) $u \in L^2(\Omega)$

(A3) $w \in L^2(\partial\Omega)$

To derive weak formulation, take test function $\varphi \in H^1(\Omega)$ integrate by parts

$$-\int_\Omega \Delta y \, \varphi \, dx = \int_\Omega \nabla y \, \nabla \varphi \, dx$$

$$\underbrace{-\int_{\partial\Omega} \frac{\partial y}{\partial \nu} \varphi \, dx}_{= \int_{\Gamma_1} \beta (y - w) \varphi \, dx} = \int_\Omega u \varphi \, dx$$

$\Longleftrightarrow$

$$\int_\Omega \nabla y \, \nabla \varphi \, dx + \beta \int_{\Gamma_1} y \varphi \, dx = \int_\Omega u \varphi \, dx + \beta \int_{\Gamma_1} w \varphi \, dx$$

$$\forall \varphi \in H^1(\Omega)$$

Now, we consider two case:

Theorem 2 For $\beta > 0$ and $w \in L^2(\partial\Omega)$ (SE2) has a unique sol. $y \in H^1(\Omega)$ and there exist a constant $c > 0$, s.th.

$$(*) \quad \|y\|_{H^1(\Omega)} \leq c \|w\|_{L^2(\partial\Omega)}$$

(a priori estimate)

Proof of $(*)$, take $\varphi = y$

Coercivity, use general. Fr. inequ.

$$\alpha \|y\|_{H^1}^2 \leq a(y,y) = \int_\Omega |\nabla y|^2 dx + \beta \int_{\partial\Omega} y^2 ds = \beta \int_{\partial\Omega} w y \, ds$$

$$\Rightarrow \quad \alpha \|y\|_{H^1(\Omega)}^2 \leq \beta \int_{\partial\Omega} w y \, ds$$

Young inequ.

$$\leq \delta\beta \int_{\partial\Omega} y^2 ds + \frac{\beta}{4\delta} \int_{\partial\Omega} w^2 ds$$

trace theorem

$$\overset{\Downarrow}{\leq} \underbrace{\delta \beta c_1}_{\alpha/2} \|y\|^2_{H^1(\Omega)} + \frac{\beta}{4\delta} \int_{\partial\Omega} w^2 \, ds$$

choose δ such that $\delta \beta c_1 = \frac{\alpha}{2}$

i.e., $\delta = \frac{\alpha}{2\beta c_1}$

$$\Rightarrow \quad \frac{\alpha}{2} \|y\|^2_{H^1(\Omega)} \leq \frac{2\beta^2 c_1}{4\alpha} \|w\|^2_{L^2(\partial\Omega)}$$

Now, we formulate 2 optimal control problems:

(CP1) (distributed control)

$$\min J(y,u) = \frac{1}{2} \int_\Omega (y - y_d)^2 dx + \frac{\gamma}{2} \int_\Omega u^2 dx$$

subj. to (SE1) $a_0(y,u) = \int_\Omega u\varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$

and $u \in \mathcal{U}_{ad} \subset L^2(\Omega)$

(CP2) (boundary control)

$$\min J(y,w) = \frac{1}{2}\int_{\Omega}(y-y_d)^2\,dx + \frac{\lambda}{2}\int_{\partial\Omega} w^2\,ds$$

s.t. (SE1) $a_1(y,\varphi) = \beta\int_{\partial\Omega} w\varphi\,ds \quad \forall \varphi \in H^1(\Omega)$

and $w \in W_{ad} \subset L^2(\partial\Omega)$

Problems to be discussed:

i) proving existence of sol. to (CP1+2)

ii) differentiability of the solution with respect to the state

iii) optimality conditions

Difficulty for proving existence:

Theorem 3: Let H be a Hilbert space

then $\dim H < \infty \iff B_1(0)$ is compact

$\rightsquigarrow$ use weak convergence.