Optimization II

Lecture 10, Feb. 9

Revision:

- **Thm. 18** (CP1) $\min J(y,u)$

s.t. (SE) $\begin{cases} -\Delta y = u & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$

and $u \in \mathcal{U}_{ad} \subset L^2(\Omega)$

has a solution $\bar{u} \in \mathcal{U}_{ad}$ $\iff$

$\exists$ an opt state $\bar{y}$ satisfying (SE), and adj. $\bar{p}$ satisfying

$$(AE) \begin{cases} -\Delta p = \gamma\,(\bar{y} - y_d) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega \end{cases}$$

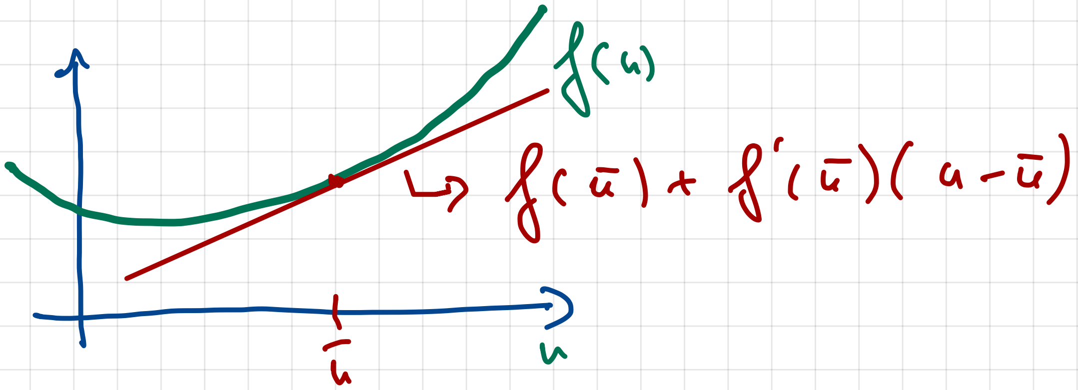
and

$$(VI) \quad (\bar{p} + \gamma \bar{u},\, u - \bar{u})_{L^2} \geq 0 \quad \forall u \in \mathcal{U}_{ad}$$

Reminder:

f is convex and diffb. $\Longrightarrow$

$$f(u) - f(\bar{u}) \geq f'(\bar{u})(u - \bar{u})$$



- Lagrangian $\mathcal{L}(y, p, u)$ can be used to derive optimality system

- box constraints

$$U_{ad}^{B} = \{ u \in L^2(\Omega) \mid \xi_0(x) \le u(x) \le \xi_1(x) \text{ a.e. in } \Omega \}$$

with $\xi_0, \xi_1 \in L^\infty(\Omega)$, $\xi_0(x) \le \xi_1(x)$ a.e.

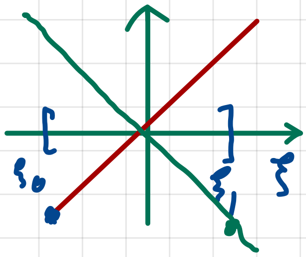
- <u>Lemma 20</u>

$$(VI) \iff \left(\bar p(x) + \gamma \bar u(x)\right)\left(\xi - \bar u(x)\right) \ge 0$$

$$\forall\, \xi \in [\xi_0(x), \xi_1(x)] \text{ a.e. in } \Omega$$

Corollary 2.1 $\bar u \in \mathcal{U}_{ad}^{B}$ be optimal

$\iff$ (1) or (2) is satisfied



(1) $$\min_{\xi \in [\xi_0(x), \xi_1(x)]} (\bar p(x) + \gamma \bar u(x))\, \xi = (\bar p(x) + \gamma \bar u(x))\, \bar u(x)$$

(weak min. principle)

(2) $$\min_{\xi \in [\xi_0(x), \xi_1(x)]} \left(p(x)\xi + \frac{\gamma}{2} \xi^2 \right) = p(x) \bar u(x) + \frac{\gamma}{2} \bar u(x)^2$$

(strong min. principle)

Proof: (1) ✓

(2) let $\bar \xi$ be sol to

$$\min_{\xi \in [\xi_0(x), \xi_1(x)]} \underbrace{p(x)\xi + \frac{\gamma}{2}\xi^2}_{g(\xi)}$$

$$\Longleftrightarrow \quad g'(\bar{\xi})(\xi - \bar{\xi}) \geq 0 \quad \forall \xi \in [\xi_0(x), \xi_1(x)]$$

$$\Longleftrightarrow \quad (p(x) + \gamma \bar{\xi})(\xi - \bar{\xi}) \geq 0$$

for $\bar{\xi} = \bar{u}(x)$ $\quad \bar{u}(x)$ sol. to (CP)

Corollary 22

$$\bar{u}(x) = \begin{cases} \xi_0(x), & \text{if } p(x) + \gamma \bar{u}(x) > 0 \\ \xi_1(x), & \text{if } \quad \text{"} \quad < 0 \\ \in [\xi_0(x), \xi_1(x)] & \text{if } \quad \text{"} \quad = 0 \end{cases}$$

Remark:

Assume $\gamma = 0$ and $p(x) \neq 0$ a.e. in Ω

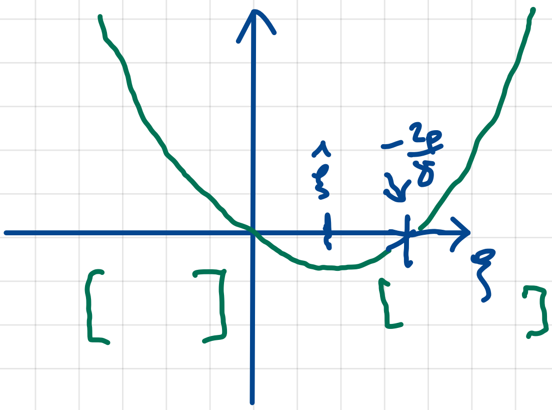
then $\bar{u}(x) = \begin{cases} \xi_0, & \text{if } p(x) > 0 \\ \xi_1, & \text{if } p(x) < 0 \end{cases}$

This means, opt. control takes values only on the boundary of U_{ad}^{B}, this is called bang-bang control.

Now, let $\gamma > 0$ and consider Corollary 21 (2).

$$\bar{\xi} = \bar{u}(x) = \arg\min_{\xi \in [\xi_0(x), \xi_1(x)]} \underbrace{\xi\left(p(x) + \frac{\gamma}{2}\xi\right)}_{g(\xi)}$$

Let $\hat{\xi}$ be unconstrained min of g



if $\hat{\xi} \in [\xi_0(x), \xi_1(x)]$

then $\bar{\xi} = \hat{\xi} = -\frac{1}{\gamma} p$

if $\hat{\xi} < \xi_0(x)$ then $\bar{\xi} = \xi_0(x)$

if $\hat{\xi} > \xi_1(x)$ then $\bar{\xi} = \xi_1(x)$

Theorem 23: Let $\bar{u} \in \mathcal{U}^B_{ad}$ be sol. to (CP1) and $\gamma > 0$, then

$$\bar{u}(x) = P_{[\xi_0(x), \xi_1(x)]}\left(-\frac{1}{\gamma} p(x)\right)$$

where $P_{[a,b]} : \mathbb{R} \to \mathbb{R}$, $x \longmapsto \min\{b, \max\{a,x\}\}$

is the projection operator onto $[a,b]$

Now, we consider the boundary control problem

$$(CP2) \quad \min \frac{1}{2} \int_{\Omega} (y - y_d)^2 dx + \frac{\gamma}{2} \int_{\partial\Omega} \omega^2 dx$$

$$\text{s.t. (SE)} \begin{cases} -\Delta y = 0 & \text{in } \Omega \\ -\frac{\partial y}{\partial \nu} = \beta(y - \omega) & \text{on } \partial\Omega \end{cases}$$

$$\omega \in W_{ad} \subset L^2(\partial\Omega)$$

Recall that (SE) in weak form is given

as $$\underbrace{\int_{\Omega} \nabla y \cdot \nabla \varphi \, dx + \beta \int_{\partial\Omega} y \varphi \, dx}_{a_1(y,\varphi)} = \beta \int_{\partial\Omega} \omega \varphi \, dx \quad \forall \varphi \in H^1(\Omega)$$

Proceed as in dealing with (CP1)

(1) LM $\Rightarrow$ for any $\omega \in W_{ad} \subset L^2(\partial\Omega)$

there is unique $y \in H^1(\Omega)$ sol. to (SE)

$\Rightarrow$ $G : \omega \longmapsto y(\omega)$, $L^2(\partial\Omega) \to H^1(\Omega)$

and Sol. operator

$$S : E_2 \circ G \,, \quad L^2(\partial\Omega) \to L^2(\Omega)$$

(2) Prove that (CP2) has a unique sol $\bar{\omega} \in W_{ad}$, analogous to the proof for (CP1)

(3) Prove F-diffb. of sol. op. $\omega \mapsto y(\omega)$

(4) Use Lagrangian to derive (AE) and (VI).

$$\mathcal{L}(y,p,w) = \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \frac{\gamma}{2}\int_{\partial\Omega} w^2 dx - \int_\Omega \nabla y \cdot \nabla p \, dx$$

$$-\beta\int_{\partial\Omega} y p \, dx + \beta\int_{\partial\Omega} w p \, dx$$

(i) derive (AE) by computing

$$\mathcal{L}_y'(\bar{y},p,\bar{u})h = 0 \implies$$

$$0 = \int_\Omega (\bar{y}-y_d)h\,dx - \int_\Omega \nabla h\cdot\nabla p\,dx - \beta\int_{\partial\Omega} h p\,dx$$

$$= \int_\Omega (\bar{y}-y_d)h\,dx + \int_\Omega h\Delta p\,dx - \int_{\partial\Omega} h\frac{\partial p}{\partial \nu}dx - \beta\int_{\partial\Omega} h p\,dx$$

$$= \int_\Omega \underbrace{\left[\bar{y}-y_d+\Delta p\right]}_{=0} h\,dx - \int_{\partial\Omega} \underbrace{\left[\frac{\partial p}{\partial\nu}+\beta p\right]}_{=0} h\,dx \quad \forall h \in H^1(\Omega)$$

for $h \in H_0^1(\Omega)$ we get

$$\implies (AE)\begin{cases} -\Delta p = \bar{y}-y_d & \text{on } \Omega \\ -\frac{\partial p}{\partial \nu} = \beta p & \text{on } \partial\Omega \end{cases}$$

(ii) derive (VI)

$$\mathcal{L}_u(\bar{y}, p, \bar{w})(w - \bar{w}) \geq 0$$

$$\mathcal{L}(y, p, w) = \frac{1}{2}\int_{\Omega}(y - y_d)^2\,dx + \frac{\gamma}{2}\int_{\partial\Omega} w^2\,dx - \int_{\Omega}\nabla y \cdot \nabla p\,dx$$

$$-\beta\int_{\partial\Omega} y p\,dx + \beta\int_{\partial\Omega} w p\,dx$$

$$\mathcal{L}_w(\bar{y}, p, \bar{w})(w - \bar{w}) = \gamma\int_{\partial\Omega}\bar{w}(w - \bar{w})\,dx + \beta\int_{\partial\Omega} p(w - \bar{w})\,dx$$

$\Rightarrow$

(VI) $$\int_{\partial\Omega}(\gamma\bar{w} + \beta p)(w - \bar{w})\,dx \geq 0 \quad \forall w \in W_{ad}$$

Theorem 24: $\bar{w} \in W_{ad} \subset L^2(\Omega)$ is sol to (CP2) $\Longleftrightarrow$

(1) $\bar{y}$ solves (SE)

(2) p solves (AE)

(3) (VI) is satisfied

Now, consider box constraints for (CP2)

$$W_{ad}^{B} = \{ w \in L^2(\partial\Omega) \mid \xi_0(x) \le w(x) \le \xi_1(x), \text{ a.e. on } \partial\Omega \}$$

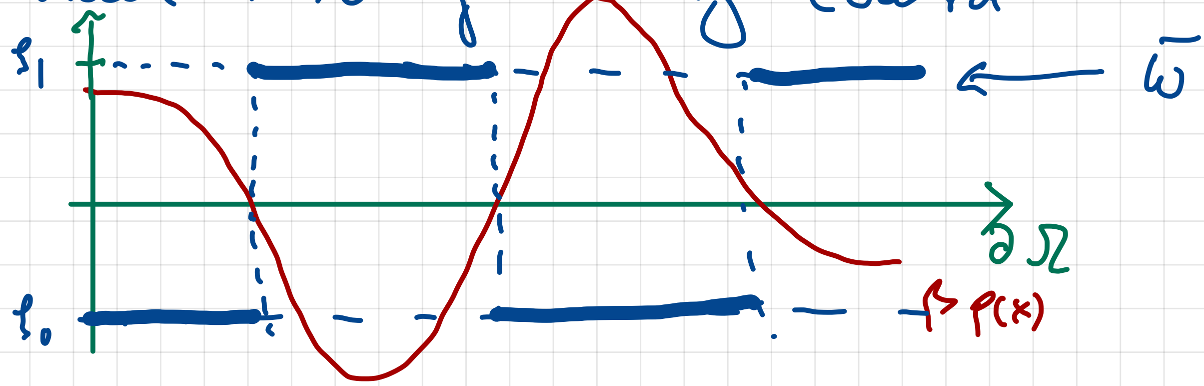
as before, we can localize the (VI) and we obtain

$$\bar{w} = \begin{cases} \xi_0(x), & \text{if } \beta p + \gamma \bar{w} > 0 \\ \xi_1(x) & \text{if } \quad -\text{"}- \quad < 0 \\ \in [\xi_0(x), \xi_1(x)], & -\text{"}- \quad = 0 \end{cases}$$

For $\gamma = 0$ and $\beta p(x) \neq 0$ a.e. on $\partial\Omega$

$$w = \begin{cases} \xi_0(x), & \text{if } \beta p(x) > 0 \\ \xi_1(x), & \text{if } \beta p(x) < 0 \end{cases},$$

we have a bang-bang control



for $\gamma > 0$, as in Theorem 2.22 we can show

$$\overline{w} = P_{[\xi_0(x), \xi_1(x)]}\left(-\frac{\beta}{\gamma}p(x)\right)$$