Weierstrass Institute for
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Leibniz Association

Optimization II
– basic facts from functional analysis

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- Notation – Multiindices

$$\alpha = (\alpha_1, ..., \alpha_n) \ , \ \alpha_i \in \mathbb{N}_0$$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \qquad , \ x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \ , \ x \in \mathbb{R}^n$$

$$D^\alpha \phi = \left(\frac{\partial}{\partial x}\right)^\alpha \phi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi$$

- Motivation: Let $y \in C^1[0,1]$ and $\phi \in C_0^\infty(0,1)$, then

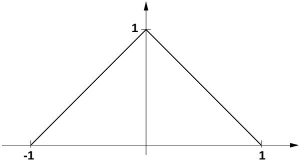
$$\int_0^1 y'\phi dx = -\int_0^1 y\phi' dx + \underbrace{y\phi\Big|_0^1}_{=0} = -\int_0^1 y\phi' dx$$

- Definition

  $y \in L^1_{loc}(\Omega)$ has a weak derivative $D^\alpha y$, if there exists a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|}\int_\Omega y(x)\phi^{(\alpha)}(x)dx \quad \text{for all } \phi \in C_0^\infty(\Omega)\,.$$

  Then $g = D^\alpha y$ is called weak derivative of y.



- Example: Let $f(x) = 1 - |x|$ then f is weakly differentiable with derivative

$$g(x) = \begin{cases} 1\,, & x < 0 \\ -1\,, & x > 0 \end{cases}.$$

- Proof:

$$\int_{-1}^{1} f(x)\phi'(x)dx = \int_{-1}^{0} (1+x)\phi'(x)dx + \int_{0}^{1} (1-x)\phi'(x)dx$$

$$= -\int_{-1}^{0} \phi(x)dx + f\phi\,\Big|_{-1}^{0} + \int_{0}^{1} \phi(x)dx + f\phi\,\Big|_{0}^{1}$$

$$= -\int_{-1}^{1} g(x)\phi(x)dx\,.$$

- Definition

$$H^m(\Omega) = \{u \in L^2(\Omega) \,|\, \partial^\alpha u \in L^2(\Omega) \quad \text{for all } \alpha \text{ with } |\alpha| \leq m\}$$

- Remark H is a Hilbert space with scalar product

$$(u,v)_{m,\Omega} = \sum_{0\leq|\alpha|\leq m} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx$$

- Important case:

$$H^1(\Omega) = \{u| \, (u,u)_{1,\Omega}^{1/2} < \infty\}$$

with scalar product

$$(u,v)_{1,\Omega} = \int_\Omega uv dx + \int_\Omega \nabla u \cdot \nabla v dx$$

and norm

$$\|u\|_{H^1(\Omega)} = (u,u)_{1,\Omega}^{1/2} = \left(\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \right)^{1/2}$$

- Theorem Let $\Omega \subset \mathbb{R}^n$ a domain with Lipschitz boundary, then there holds
  - $H^s(\Omega) \subset C^k(\bar{\Omega})$, if $s > k + \frac{n}{2}$.
  - For $s > k + \frac{n}{2}$ there exists a constant $C > 0$, such that

$$\|u\|_{C^k(\bar{\Omega})} \leq C\|u\|_{H^s(\Omega)} \quad \text{(continuous embedding)}\,.$$

  - The embedding is even compact.
- Examples
  - $n = 1$, i.e., $\Omega = (a, b)$, $s = 1 > 0 + \frac{1}{2}$ then

$$H^1(a, b) \subset C[a, b]$$

  - $n = 2$, $s = 1 \not> 0 + 1$ no continuous embedding of $H^1(\Omega)$ in $C(\bar{\Omega})$.
  - $n = 3$, $s = 2 > 0 + \frac{3}{2}$, hence $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$.

- Theorem

  Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, then there exists a linear, surjective, continuous mapping (the trace operator)

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

with

$$Tu = u\big|_{\partial\Omega}$$

and there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)} .$$

- Important case:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \,\big|\, T(v) = v\big|_{\partial\Omega} = 0\}$$

- **Lemma** Let Ω be an open Lipschitz domain and $\Gamma_1 \subset \partial\Omega$ measurable with $|\Gamma_1| > 0$. Then there exists a constant c independent of $y \in H^1(\Omega)$, such that

$$\|y\|^2_{H^1(\Omega)} \leq c \left(\int_\Omega |\nabla y|^2 dx + \int_{\Gamma_1} y^2 ds \right).$$

- Theorem

  Let $\Omega \subset \mathbb{R}^n$ be a ein Lipschitz domain and $u, v \in H^1(\Omega)$. Then, there holds

$$\int_\Omega u(x)\partial_i v(x)dx = -\int_\Omega \partial_i u(x)v(x)dx + \int_{\partial\Omega} u(s)v(s)\nu_i(s)ds$$

  for $1 \leq i \leq n$, where ν_i is the i-th component of the outer normal ν.

- Definition Let $a(.,.) : V \times V \to \mathbb{R}$ be a bilinear form on a normed linear space V.
  - It is called bounded (or continuous), if there exists a $C > 0$ such that

$$|a(v,w)| \leq C\|v\|_V\|w\|_V \quad \text{for all } \; v,w \in V$$

  - It is called corecive, if there exists an $\alpha > 0$ such that

$$a(v,v) \geq \alpha\|v\|_V^2 \quad \text{for all } \; v \in V \; .$$

- Theorem (Lax-Milgram Let $(V,(.,.))$ be a Hilbert space, $a(.,.)$ a coercive and continuous bilinear form and $F \in V^*$ (i.e., a linear, continuous mapping $V \to \mathbb{R}$), then there exists one and only one $y \in V$ such that

$$a(y,v) = F(v) \quad \text{für alle } \; v \in V \; .$$

- Equivalence of boundedness and continuity
  A linear operator between normed spaces is bounded if and only if it is continuous.

- Riesz Representation Theorem If X is a Hilbert space, then

$$J(x)(y) := (y,x)_X \quad \text{for } x, y \in X$$

defines an isometric linear isomorphism $J : X \to X'$.

- Young's Inequality

For $\delta > 0$ and real numbers a, b, there holds

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2$$

- Hölder's Inequality

Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, then there holds

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$