Rothe method for parabolic equations

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March, 2022

- assume $\Omega \in \mathbb{R}^n$ with smooth boundary (at least $C^{1,1}$)
- consider

$$y_t - \operatorname{div}(k(x)\operatorname{grad} y) = \alpha u \quad \text{in } \Omega \times (0,T)\,. \tag{1}$$

$$\frac{\partial y}{\partial \nu} = 0 \quad \text{in } \partial\Omega \times (0,T)\,. \tag{2}$$

$$y(0) = y_0 \quad \text{in } \Omega\,. \tag{3}$$

- $T < \infty$ end-time
- we understand (1)–(3) as an evolution equation in a Hilbert space
- Definition
  Let $I \subset \mathbb{R}$ an interval and H Hilbert space, then $C(I,H)$ is the space of all continuous functions $f : I \to H$, $t \mapsto f(t) \in H$ with norm
  $\|u\| = \sup_{t \in I} \|u(t)\|_H$.

- $f \in C(I, H)$ not necessarily continuous in space, consider e.g., $g \in C([0, T])$ and $u \in L^2(\Omega)$, then $f(t) = g(t)u$ not continuous in x.
- thorough introduction see, e.g., Zeidler: Nonlinear FA, Vol. IIa, Springer, short intro in Evan's PDE book
- general idea: analogous to introduction of $L^p(\Omega)$, introduce $L^p(I, H)$ as completion of $C(I, H)$ with respect to norm

$$\|f\|_{L^p(0,T;\,H)} = \left(\int_I \|f(t)\|_H^p dt \right)^{1/p} .$$

(1) $C([0,T];\, L^2(\Omega))$ with norm $\|f\| = \sup_{[0,T]} \left(\int_\Omega f(t)^2 dt\right)^{1/2}$

(2) $L^2((0,T);\, L^2(\Omega))$ with norm $\|f\| = \left[\int_0^T \left(\int_\Omega f^2(x,t)dx\right) dt\right]^{1/2}$

(3) $H^1((0,T);\, L^2(\Omega))$ with norm

$$\begin{aligned}\|f\| &= \left(\int_0^T \left(\|f\|^2_{L^2(\Omega)} + \|f_t\|^2_{L^2(\Omega)}\right) dt\right)^{1/2} \\ &= \left(\int_0^T \int_\Omega \left(f^2(x,t) + f_t^2(x,t)\right) dxdt\right)^{1/2}.\end{aligned}$$

(4) $L^2((0,T;\, H^1(\Omega))$ with norm

$$\|f\| = \left[\int_0^T \int_\Omega \left(f^2(x,t) + |\nabla f(x,t)|^2\right) dxdt\right]^{1/2}.$$

- Weak formulation

$$\int_\Omega y_t(t)\varphi dx + \underbrace{k\int_\Omega \nabla y(t)\nabla\varphi dx}_{=a(y(t),\varphi)} = \underbrace{\int_\Omega \alpha u\varphi dx}_{=F(\varphi)}$$

or

$$(y_t(t),\varphi)_{L^2(\Omega)} + a(y(t),\varphi) = F(\varphi) \quad \text{for all } \varphi\in H^1(\Omega), t\in(0,T) \tag{4}$$

$$y(0) = y_0 \quad \text{in } \Omega\,. \tag{5}$$

- we will show, (4),(5) has a unique sol.

$$y\in H^{1,1}(Q) := H^1(0,T;L^2(\Omega))\cap L^2(0,T;\ H^1(\Omega))$$

where $Q=\Omega\times(0,T)$ is the space-time cylinder

- Lemma: Let $y\in H^{1,1}(Q)$ then $y\in C([0,T];L^2(\Omega))$.
- Hence initial condition (5) is well-defined

- two strategies to prove existence of a weak solution
  - Galerkin ansatz
    Replace $V = H^1(\Omega)$ by finite dimensional approximation V_m, put $y^m(t) = \sum_{i=1}^m g_i(t)\varphi_i$, where φ_i is a basis of V_m. For m fixed, y^m is given as the solution to an ODE, passing to the limit gives solution to (4)–(5), see, e.g. Evan's book.
  - Rothe method (which we will follow)
    replace $y_t(t)$ by difference quotient $\frac{y(t+h)-y(t)}{h}$.
- Lemma:
  Let $f \in L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 < p \leq \infty$. There is a constant $C > 0$,s.t. for all $\Omega' \subset\subset \Omega$ and $h < \operatorname{dist}(\Omega', \partial\Omega)$ there holds

$$D_i^h f := \frac{f(x + he_i) - f(x)}{h} \in L^p(\Omega') \quad \text{with} \quad \|D_i^h f\|_{L^p(\Omega')} \leq C\,.$$

  Then, $u_{x_i} \in L^p(\Omega)$ and

$$\|u_{x_i}\|_{L^p(\Omega)} \leq C\,.$$

- Assumptions on data
  (A1) $k > 0$ konstant
  (A2) $\alpha \in L^\infty(Q)$ mit $Q = \Omega \times (0,T)$ und $\|\alpha\|_{L^\infty(Q)} \leq \bar{\alpha}$
  (A3) $u \in L^2(Q)$, $y_0 \in H^1(\Omega)$.
- Theorem: Assume $(\mathrm{A1})$–$(\mathrm{A3})$, then (4)–(5) has a unique solution $y \in H^{1,1}(Q)$.
- Proof in 4 steps
  1. uniqueness via apriori estimate
  2. time discretization, solution of elliptic problems
  3. discrete apriori estimate
  4. passing to the limit

Let y be a sol. to (4)–(5). Insert $\varphi = y_t$ in (4) integrate with respect to time

$$\int_0^t \int_\Omega y_\xi^2 \, dx d\xi + \underbrace{k \int_0^t \int_\Omega \nabla y \nabla y_\xi \, dx d\xi}_{I_1} = \underbrace{\int_0^t \int_\Omega \alpha u y_\xi \, dx d\xi}_{I_2} .$$

for the integrals we obtain

$$\begin{aligned} I_1 &= \frac{k}{2} \int_0^t \int_\Omega \frac{d}{d\xi} |\nabla y|^2 dx d\xi = \frac{k}{2}, \int_\Omega \int_0^t \frac{d}{d\xi} |\nabla y|^2 d\xi dx \\ &= \frac{k}{2} \int_\Omega |\nabla y(t)|^2 dx - \frac{k}{2} \int_\Omega |\nabla y_0|^2 dx . \end{aligned}$$

$$I_2 \le \frac{1}{2} \int_0^t \int_\Omega y_\xi^2 dx d\xi + \frac{\bar{\alpha}^2}{2} \int_0^t \int_\Omega u^2 dx ds .$$

Altogether, we get

$$\frac{1}{2}\int_0^t\int_\Omega y_\xi^2\,dxd\xi+\frac{k}{2}\int_\Omega|\nabla y(t)|^2dx \\ \leq\frac{\bar{\alpha}^2}{2}\int_0^t\int_\Omega u^s\,dxd\xi+\frac{k}{2}\int_\Omega|\nabla y_0|^2dx\,. \qquad (6)$$

Remark: Functions $y\in H^1(0,T;\,L^2(\Omega))$ are absolutely continuous, i.e.

$$y(t)=y(0)+\int_0^t y_\xi d\xi \quad \text{f.ü. in } \Omega$$

Using Hölder inequality and $(a+b)^2\leq 2a^2+2b^2$ we get

$$\int_\Omega y(t)^2dx=\int_\Omega\left(y_0+\int_0^t y_\xi d\xi\right)^2 x\leq 2\int_\Omega y_0^2dx+2T\int_0^t\int_\Omega y_\xi^2dxd\xi \qquad (7)$$

hence, from (6) we get

$$\|y\|^2_{L^\infty(0,T;\,H^1(\Omega))}+\|y\|^2_{H^1(0,T;\,L^2(\Omega))}\leq C\left(\|u\|^2_{L^2(0,T;\,L^2(\Omega))}+\|y_0\|^2_{L^2(\Omega)}\right).$$

Since (4) is linear, we can infer uniqueness.

- Leti $M \in \mathbb{N}$ fixed and $h = \frac{T}{M}$, $y^0 := y_0$. For $m \in \{1, ..., M\}$ we define

$$u_\alpha^m(x) = \frac{1}{h} \int_{(m-1)h}^{mh} \alpha(x,t) u(x,t) dt \,. \tag{8}$$

- We replace $y_t(mh)$ by $\delta_h y^m := \frac{1}{h}\left(y^m - y^{m-1}\right)$ and consider an implicit time-discrete scheme

$$\int_\Omega \delta_h y^m \varphi dx + k \int_\Omega \nabla y^m \nabla \varphi dx = \int_\Omega u_\alpha^m \varphi dx \quad \text{for all } \varphi \in H^1(\Omega) \,. \tag{9}$$

- applying Lax-Milgram), inductively we get unique sol. $y^m \in H^1(\Omega)$ for $m = 1, ..., M$ of elliptic problems (9).

We derive time discrete counterpart of (6). Insert $\varphi = y^m - y^{m-1}$ to get

$$\int_\Omega \delta_h y^m (y^m - y^{m-1}) dx + k \int_\Omega \nabla y^m \nabla (y^m - y^{m-1}) dx = \int_\Omega u_\alpha^m (y^m - y^{m-1}) dx\,.$$

Summing up for $m = 1$ to l yields

$$\sum_{m=1}^{l} h \int_\Omega (\delta_h y^m)^2 dx + \underbrace{k \sum_{m=1}^{l} \int_\Omega \nabla y^m \nabla (y^m - y^{m-1}) dx}_{I_2} = \underbrace{\sum_{m=1}^{l} \int_\Omega u_\alpha^m (y^m - y^{m-1}) dx}_{I_1}$$

We apply inequalities of Hölder und Young to I_1::

$$I_1 \le \sum_{m=1}^{l} h \int_\Omega u_\alpha^m \delta_h y^m dx \le \frac{1}{2} \sum_{m=1}^{l} h \int_\Omega (u_\alpha^m)^2 dx + \frac{1}{2} \sum_{m=1}^{l} h \int_\Omega (\delta_h y^m)^2 dx$$

$$\le \frac{\bar{\alpha}^2}{2} \int_0^T \int_\Omega u^2 dx dt + \frac{1}{2} \sum_{m=1}^{l} h \int_\Omega (\delta_h y^m)^2 dx$$

Here, we have used

$$\sum_{m=1}^{l} h \int_\Omega (u_\alpha^m)^2 dx = \sum_{m=1}^{l} h \frac{1}{h^2} \int_\Omega \left(\int_{(m-1)h}^{mh} \alpha(x,t) u(x,t) dt \right)^2 dx$$

$$\overset{\text{Hölder}}{\leq} \sum_{m=1}^{l} \int_{(m-1)h}^{mh} \int_\Omega \alpha^2 u^2 dx dt \leq \bar{\alpha}^2 \int_0^T \int_\Omega u^2 dx dt \, .$$

Next, we apply the identity $|a|^2 - a \cdot b = \frac{1}{2} |a-b|^2 + \frac{1}{2} |a|^2 - \frac{1}{2} |b|^2$ to I_2,

$$\sum_{m=1}^{l} \int_\Omega \nabla y^m \nabla (y^m - y^{m-1}) dx = \sum_{m=1}^{l} \left(\int_\Omega |\nabla y^m|^2 dx - \int_\Omega \nabla y^m \nabla y^{m-1} \right)$$

$$= \frac{1}{2} \sum_{m=1}^{l} \int_\Omega |\nabla y^m - y^{m-1}|^2 dx + \frac{1}{2} \sum_{m=1}^{l} \int_\Omega (|\nabla y^m|^2 - |\nabla y^{m-1}|^2) dx$$

$$= \frac{1}{2} \sum_{m=1}^{l} \int_\Omega |\nabla y^m - \nabla y^{m-1}|^2 dx + \frac{1}{2} \int_\Omega |\nabla y^l|^2 dx - \frac{1}{2} \int_\Omega |\nabla y^0|^2 dx \, .$$

All in all, we obtain

$$\frac{1}{2}\sum_{m=1}^{M} h\|\delta_h y^m\|_{L^2(\Omega)}^2 + \frac{k}{2}\max_{1\le l\le M}\|\,|\nabla y^m|\,\|_{L^2(\Omega)}^2$$

$$+\frac{k}{2}\sum_{m=1}^{l}\int_\Omega |\nabla y^m - \nabla y^{m-1}|^2 dx \le \frac{\bar{\alpha}^2}{2}\int_0^T\int_\Omega u^2 dx dt + \frac{k}{2}\int_\Omega |\nabla y_0|^2 dx\,. \tag{10}$$

Define linear and constant in time interpolating functions

$$y_h(t) = y^m + \frac{t - mh}{h} \left(y^m - y^{m-1}\right) \quad \text{for } t \in [(m-1)h, mh]$$

$$\hat{y}_h(t) = y^m \text{, for } t \in [(m-1)h, mh]$$

From (10) we get

$$\|y_{h,t}\|_{L^2(0,T;\,L^2(\Omega))} + \|\,|\nabla \hat{y}_h|\,\|_{L^2(\Omega)} \leq C_1\,.$$

Moreover, as for the time-continuous case, see (7), we obtain

$$\|y_h(t)\|^2_{L^2(\Omega)} \leq 2\int_\Omega y_0^2 dx + 2T \sum_{m=1}^{M} \int_{(m-1)h}^{mh} \int_\Omega (y_{h,t})^2 dx dt$$

$$\leq 2\int_\Omega y_0^2 dx + 2T \sum_{m=1}^{M} h \int_\Omega (\delta_h y^m)^2 dx \leq C_2\,.$$

Using triangle inequality, we get $\|\nabla y_h(t)\|_{L^2(\Omega)} \leq C_3$,
altogether, we have

$$\|y_h\|_{H^1(0,T;\,L^2(\Omega))\cap L^\infty(0,T;\,H^1(\Omega))} + \|\hat{y}_h\|_{L^\infty(0,T;\,H^1(\Omega))} \leq C_4\,. \tag{11}$$

From (10) we infer

$$\|\nabla y_h - \nabla \hat{y}_h\|^2_{L^2(0,T;\,L^2(\Omega))} = \int_0^T \int\limits_\Omega |\nabla y_h - \nabla \hat{y}_h|^2 dxdt$$

$$= \sum_{m=1}^{M} \int\limits_{(m-1)h}^{mh} \int\limits_\Omega \left(\nabla y^m + \tfrac{t-mh}{h}\left(\nabla y^m - \nabla y^{m-1}\right) - \nabla y^m\right)^2 dxdt$$

$$= \sum_{m=1}^{M} \int\limits_{(m-1)h}^{mh} \left(\tfrac{t-mh}{h}\right)^2 dt \int\limits_\Omega \left|\nabla y^m - \nabla y^{m-1}\right|^2 dx$$

$$= \frac{h}{3} \sum_{m=1}^{M} \int\limits_\Omega |\nabla y^m - \nabla y^{m-1}|^2 dx \underset{h\to 0}{\longrightarrow} 0\,. \tag{12}$$

Utilizing (11) we get

$$\begin{aligned} y_h \longrightarrow y \quad & \text{schwach}^* \text{ in } L^\infty(0,T;\, H^1(\Omega)) \\ & \text{schwach in } H^1(0,T;\, L^2(\Omega)) \cap L^2(0,T;\, H^1(\Omega)) \end{aligned} \tag{13}$$

$$\begin{aligned} \hat{y}_h \longrightarrow \hat{y} \quad & \text{schwach}^* \text{ in } L^\infty(0,T;\, H^1(\Omega)) \\ & \text{schwach in } L^2(0,T;\, H^1(\Omega))\,. \end{aligned} \tag{14}$$

Moreover, we observe

$$\int_0^T \int_\Omega (y_h - \hat{y}_h)^2 dx dt = \sum_{m=1}^M \int_{(m-1)h}^{mh} \int_\Omega \frac{(t-mh)^2}{h^2} (y^m - y^{m-1})^2 dx dt$$

$$= \frac{h^2}{3} \sum_{m=1}^M h \int_\Omega (\delta_h y^m)^2 dx \longrightarrow 0\,.$$

and together with (12)

$$\|y_h - \hat{y}_h\|^2_{L^2(0,T;\,H^1(\Omega))} \xrightarrow[h\to 0]{} 0, \quad \text{i.e.,} \quad y = \hat{y} \quad \text{a.e. in } Q\,.$$

Now, we get back to (4). Multiplying with $\psi \in L^2(0,T)$ with test function $\phi = \psi\varphi \in L^2(Q)$ we get

$$\int_0^T \int_\Omega y_{h,t}\phi \, dxdt + k \int_0^T \int_\Omega \nabla \hat{y}_h \nabla \phi \, dxdt = \int_0^T \int_\Omega \hat{u}_\alpha^h \phi \, dxdt$$

where $\hat{u}_\alpha^h(t) = u_\alpha^m(x)$ for $t \in ((m-1)h, mh]$.
There holds

$$\hat{u}_\alpha^h \to \alpha u \quad \text{strongly in } \; L^2(Q) \, .$$

Utilizing (13), (14) and $y = \hat{y}$ a.e. in Q we obtain

$$\int_0^T \left(\int_\Omega y_t \varphi dx + k \int_\Omega \nabla y \nabla \varphi dx - \int_\Omega \alpha u \phi dx \right) \psi(t) dt = 0 \quad \text{for all } \; \psi \in L^2(0,T)$$

Thus, y solves (4)–(5).