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PDE-constrained optimal control

Problem sheet 2

1. Consider Problem (CP2) for optimal boundary control:

$$\min_{w \in W_{ad}} J(y, w)$$

subject to

$$\begin{aligned} -\Delta y &= f && \text{in } \Omega \\ -\frac{\partial y}{\partial \nu} &= \beta(x)(y - w) && \text{in } \partial\Omega \end{aligned}$$

where

$$J(y, w) = \frac{1}{2} \int\limits_{\Omega} (y - y_d)^2 \, dx + \frac{\gamma}{2} \int\limits_{\partial\Omega} w^2 ds.$$

a) Derive the weak formulation of the state equation and use the Lax-Milgram Lemma to show that it has a unique solution specifying assumptions on w and β.

b) Show that (CP2) admits a unique solution under suitable assumptions on W_{ad} (which ones?).

c) Rigorously derive the necessary (and sufficient) first order optimality conditions.

2. Consider the optimal control problem

$$\min J(y, u) = \frac{1}{2} \int_{\Omega} (y - y_d)^2 dx + \int_{\Gamma} e_{\Gamma} y ds + \frac{1}{2} \int_{\Omega} u^2 dx.$$

$$\begin{aligned} \text{subject to} \quad -\Delta y + y &= u + e_{\Omega} && \text{in } \Omega \\ \frac{\partial y}{\partial \nu} &= 0 && \text{in } \partial\Omega \\ \text{and} \quad 0 &\leq u(x) \leq 1 && \text{a.e. in } \Omega \end{aligned}$$

a) Use the Lagrangean to formally derive the optimality system.

b) Compute the adjoint q, e_Ω and e_Γ such that the optimal state is $\bar{y} \equiv 1$ and the optimal control is the cut-off rotational paraboloid

$$\bar{u}(x) = \mathcal{P}_{[0,1]}\left\{\left(12r^2 - \frac{1}{3}\right)\right\},$$

with apex $\hat{x} = \left(\frac{1}{2}, \frac{1}{2}\right)^T$ and

$$r(x) = |x - \hat{x}| = \sqrt{\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2}.$$

3. Consider the simplified supraconductivity model

$$\min_{u \in U_{ad}} \frac{1}{2}\int_\Omega (y - y_d)^2\, dx + \frac{\gamma}{2}\int_\Omega u^2 ds$$

subject to

$$\begin{aligned} -\Delta y + y + y^3 &= f \qquad \text{in } \Omega \\ \frac{\partial y}{\partial \nu} &= 0 \qquad \text{in } \partial\Omega \\ \text{and } -2 \leq u(x) &\leq 2 \ \text{ a.e. in } \Omega \end{aligned}$$

a) Derive formally the necessary optimality conditions using the Lagrangean.

b) Let $\gamma = 1$ and $y_d = 9$. Derive an explicit expression for the optimal state $\bar{y}$ and the adjoint p such that $\bar{u} \equiv 2$ is the optimal control.