Optimization II

Lecture 6

Thursday, Feb. 3

Theorem 3.11 Let $y_d \in L^2(\Omega)$, $\gamma \geq 0$, and $\mathcal{U}_{ad} \subset L^2(\Omega)$, closed, convex and bounded. Then, (CP) has a unique sol. $\bar{u} \in \mathcal{U}_{ad}$.

Proof: (i) Existence of an opt. control

Let $$\ell = \inf_{u \in \mathcal{U}_{ad}} J(y(u), u)$$

and $\{u_n\} \subset \mathcal{U}_{ad}$ s.th.

$$J(y(u_n), u_n) \longrightarrow \ell$$

$\mathcal{U}_{ad}$ is bounded

$\overset{\text{Lemma 3.10}}{\Longrightarrow}$ $\exists$ subsequence $\{u_{n'}\}$

$$u_{n'} \longrightarrow \bar{u} \quad \text{weakly in } L^2(\Omega)$$

$\overset{\text{Lemma 3.8}}{\Longrightarrow}$ $\bar{u} \in \mathcal{U}_{ad}$

Let $y_n = S(u_n)$ sol. to (SE), i.e.

$$\int_\Omega \nabla y_n \cdot \nabla \varphi \, dx = \int_\Omega u_n \cdot \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega)$$

Take $\varphi = y_n$ and derive a priori estimate:

$$\|y_n\|^2_{H^1(\Omega)} \leq \int_\Omega |\nabla y_n|^2 dx = \int_\Omega u_n y_n \, dx$$

$$\leq \|u_n\|_{L^2(\Omega)} \underbrace{\|y_n\|_{L^2(\Omega)}}_{\leq \|y_n\|_{H^1(\Omega)}}$$

$$\Rightarrow \|y_n\|_{H^1(\Omega)} \leq \|u_n\|_{L^2(\Omega)} \leq C$$

C indep. of n.

Lemma 3.16 $\Rightarrow$ $\exists$ subsequ. $\{y_{n_k}\}$

s.t. $y_{n''} \to \bar{y}$ in $H_0^1(\Omega)$

now, we show that $\bar{y} = S(\bar{u})$

to this end, go back to weak form.

$$\int_\Omega \nabla y'' \cdot \nabla \varphi \, dx = \int_\Omega u_{n''} \varphi \, dx$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\int_\Omega \nabla \bar{y} \cdot \nabla \varphi \, dx = \int_\Omega \bar{u} \varphi \, dx \qquad \forall \varphi \in H_0^1(\Omega)$$

$(LU) \Rightarrow \quad \bar{y} = S\bar{u}$

$J(y(u),u)$ is convex

lemma 3.9

$\Rightarrow$

$$f(\bar{u}) \leq \liminf_{n'' \to \infty} f(u_{n''}) = \ell$$

$$= \inf J(y(u),u)$$

$\Rightarrow$ $\bar{u}$ is an optimal control

with optimal state $\bar{y} = S(\bar{u})$

(ii) uniqueness

we know,

$$(*) \qquad \left[\frac{1}{2}(a+b)\right]^2 < \frac{1}{2}a^2 + \frac{1}{2}b^2$$

for $a \neq b$

assume $\bar{u}_1 \neq \bar{u}_2 \in \mathcal{U}_{ad}$ are sol. to (CP1) and define

$$\bar{u} = \frac{1}{2}\bar{u}_1 + \frac{1}{2}\bar{u}_2 \quad \text{then}$$

$$f(\bar{u}) = \frac{1}{2}\int_\Omega \left(S\left(\frac{1}{2}\bar{u}_1 + \frac{1}{2}\bar{u}_2\right) - y_d\right)^2 dx$$

$$+ \frac{\lambda}{2}\int_\Omega \left(\frac{\bar{u}_1}{2} + \frac{\bar{u}_2}{2}\right)^2 dx$$

S linear

$$= \frac{1}{2}\int_\Omega \left[\frac{1}{2}\left(S(u_1) - y_d\right) + \frac{1}{2}\left(S(u_2) - y_d\right)\right]^2$$

$$+\frac{\gamma}{2}\int_\Omega \left(\frac{\bar{u}_1}{2}+\frac{\bar{u}_2}{2}\right)^2 dx$$

(*)
$$< \frac{1}{2}\underbrace{f(\bar{u}_1)}_{=\ell} + \frac{1}{2}\underbrace{f(\bar{u}_2)}_{=\ell} = \ell$$

contradiction $\Rightarrow$ $\bar{u}_1 = \bar{u}_2$.

<u>Remarks:</u>

(1) The assertion remains true, if $U_{ad} = L^2(\Omega)$ and $\gamma > 0$, then there holds

(*) $$J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \frac{\gamma}{2}\int_\Omega u^2 dx$$

$$\geq \frac{\gamma}{2}\int_\Omega u^2 dx = \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)}$$

In the proof, take minimizing sequence then bound for $\{u_n\}$

comes from (*).

(2) Uniqueness can be proven for any strictly convex cost functional.

<u>Corollary 3.12</u>

Let $W_{ad} \subset L^2(\partial\Omega)$ closed, bounded and convex. Then, (CP2) has a unique solution $\bar{w} \in W_{ad}$.